The shifted convolution problem in function fields

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The functions $d(n)$ and $d_k(n)$

Recall that for $\Re(s) > 1$ and $k > 0$, one defines

$$\zeta(s)^k = \left(\sum_{n \geqslant 1} \frac{1}{n^s} \right)^k = \sum_{n \geqslant 1} \frac{d_k(n)}{n^s}.$$

If k is a positive integer, then $d_k(n)$ is the number of ways to write n as a product of the form $n_1 \cdots n_k$, and one usually writes $d_2(n) = d(n)$.

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$$\sum_{n \leqslant x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x)$$

where

$$\Delta(x) = \begin{cases} O(x^{1/2}) & \text{(Dirichlet, 1849)} \\ O(x^{1/3+\epsilon}) & \text{(Voronoi, 1903)} \\ \Omega_{\pm}(x^{1/4}) & \text{(Hardy, 1915)} \\ \vdots & \text{(van der Corput, Kolesnik,} \\ \vdots & \text{Iwaniec-Mozzochi, Huxley)} \\ O(x^{131/416+\epsilon}) & \text{(Huxley, 2003)} \\ O(x^{0.314483\cdots+\epsilon}) & \text{(Li–Yang, 2023)} \end{cases}$$

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Estermann's results uses cancellation in Kloosterman sums, as do all subsequent results.

General divisor correlations

More generally, one can look at

$$\sum_{n \leqslant x} d_k(n) d_\ell(n+h),$$

for $k, \ell \geqslant 2$. There are some results in the literature (most recently due to Topacogullari) for $\ell = 2$, $k \geqslant 3$, but even the leading order behaviour when $h = 1$ and $k, \ell \geqslant 3$ is not known rigorously.

Connection to moments

The case $\ell = k$ is of particular importance due to the connection, first observed by Atkinson, to moments

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k} dt.$$

$$|\zeta|^{2k} = \zeta^k \overline{\zeta}^k \approx \sum_{m,n \ll T^{k/2}} \frac{d_k(n) d_k(m)}{(mn)^{1/2}} \times \left(\frac{m}{n}\right)^{it}.$$

Integrating over $t \in [T, 2T]$ essentially restricts

$$|m - n| \ll T^{k/2-1+\epsilon}.$$

Setting $m = n + h$, this basically becomes a weighted shifted convolution

$$\sum_{h \ll T^{k/2-1+\epsilon}} (\cdots) \sum_{n \ll T^{k/2}} d_k(n) d_k(n+h).$$

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$d(n) = \sum_{\substack{m \mid n \\ m>0}} 1$	$\leftrightarrow$	$d(f) = \sum_{\substack{g \mid f \\ g \in \mathcal{M}}} 1$

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- Fix q, and let $n \to \infty$. Conrey & Florea (unpublished), Gorodetsky (2021), Woo (2023), Yiasemides (2024): if $\deg(h) < n$,

$$\frac{1}{q^n} \sum_{f \in \mathcal{M}_n} d(f)d(f+h) = P_{2,h}(n).$$

Theorem (Florea, L, Malik, and Sahay (2025+))

Let $h \in \mathcal{M}$ *and* $d = \max\{\deg(h), n\}$. *Then*

$$\sum_{f \in \mathcal{M}_n} d(f)d(f+h) = q^n \sum_{\substack{g|h \\ \deg(g) \le [n/2]}} \frac{1}{|g|} \Big[4\deg(g)^2 \Big(1 - \frac{1}{q}\Big) - 2\deg(g)\Big(2 + \frac{2}{q} + \Big(1 - \frac{1}{q}\Big)(n+d)\Big) + \Big[2n + 2 + (d-1)\Big(n\Big(1 - \frac{1}{q}\Big) + 1 + \frac{1}{q}\Big)\Big]\Big] + E,$$

where $E = 0$ *if* $\deg(h) \le n+1$ *or* $\deg(h) = n+2$ *and* n *is odd; otherwise*

$$E \ll q^{\deg(h)/2 + \varepsilon \deg(h)}.$$

Gives an asymptotic formula when $\deg(h) < (2-\varepsilon)n$. Compare to Meurman!

The general idea - duality of divisors

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Using additive characters to expand,

$$= 2\sum_{\deg(g)\leqslant\frac{1}{2}\max\{n,\deg(h)\}} \sum_{f\in\mathcal{M}_n} d(f)\frac{1}{|g|}\sum_{a \pmod g} e\Big(\frac{a(f+h)}{g}\Big).$$

Voronoi summation

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with uniformity in g. The main new ingredient is Voronoi summation formula, which will convert this into sums of the type

$$\sum_{f \leqslant \mathcal{M}_\nu} d(f) e\Big(\frac{\overline{a}f}{g}\Big),$$

with $\nu \approx 2\deg(g) - n$.

Theorem

Let $g \in \mathcal{M}_{\leqslant n/2}$ and $c \in \mathbb{F}_q[T]$ be such that $(a, g) = 1$. Then,

$$\sum_{f \in \mathcal{M}_n} d(f) e\Big(\frac{af}{g}\Big) = \frac{q^n}{|g|}(n + 1 - 2\deg(g))$$
$$+ \frac{q^n}{|g|} \sum_{\lambda \in \mathbb{F}_q^\times} \sum_{\eta_1, \eta_2 \in \{0,1\}} \sum_{0 \leqslant k \leqslant \nu} b_{\nu-k}(\lambda, \eta_1, \eta_2) \sum_{f \in \mathcal{M}_k} d(f) e\Big(\frac{\lambda \bar{a} f}{g}\Big),$$

where $\nu = 2\deg(g) - n - 2$.

A functional equation for the Estermann function

Let

$$\mathcal{D}_2(u, c/g) = \sum_{f \in \mathcal{M}} d(f) e(cf/g) u^{\deg f}.$$

Further, let $g \in \mathcal{M}_{\geq 1}$ and $c \in \mathbb{F}_q[T]$ be such that $(c, g) = 1$. Then, for $u \in \mathbb{C}$,

$$\mathcal{D}_2\left(u; \frac{c}{g}\right) = (qu^2)^{\deg(g)-1} \sum_{\lambda \in \mathbb{F}_q^{\times}} \mathcal{A}_\lambda(u) \mathcal{D}_2\left(\frac{1}{qu}; \frac{\lambda \bar{c}}{g}\right),$$

where

$$\mathcal{A}_\lambda(u) = \frac{\mathrm{Kl}_2(-1, \lambda)}{q} + \frac{1 + q - 2qu}{q(1 - qu)^2}$$

and

$$\mathrm{Kl}_2(\alpha, \beta) = \sum_{\gamma \in \mathbb{F}_q^{\times}} e_q(\alpha\gamma + \beta\gamma^{-1}).$$

Some remarks about the results

- The Voronoi formula will generate Kloosterman sums over $\mathbb{F}_q[T]$,

$$S(f, h, g) = \sum_{a \pmod{g}} e\Big(\frac{af + \bar{a}h}{g}\Big).$$

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- The functional equation of $\mathcal{D}_2$ ultimately relies on functional equations for the Hurwitz zeta function. This latter functional equation can be proved by bare hands (it only requires orthogonality of additive characters over $\mathbb{F}_q$!).

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- The results generalize to convolutions of 1 with Dirichlet characters χ of (for simplicity) prime modulus,

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When χ is quadratic, r_χ is the norm-counting function of a quadratic extension of $\mathbb{F}_q[T]$. Compare to the sums-of-squares function

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We also prove results about correlations of these.

Thank You!