# MOMENTS OF $\zeta(s,\alpha)$ ON THE CRITICAL LINE

(PARTLY JOINT W/ WINSTON HEAP)

OUTLINE :

§1. INTRODUCTION $\zeta(s,\alpha)$

§2. MOMENTS OF $\zeta(s)$ (EXPOSITORY)

§3. MOMENTS OF $\zeta(s;a/q)$ (THESIS)

§4. MOMENTS OF $\zeta(s,\alpha)$, $\alpha \notin \mathbb{Q}$ (W/ HEAP)

## §1 INTRODUCTION:

$$\zeta(s,\alpha) = \sum_{n\geq 0} \frac{1}{(n+\alpha)^s}$$

$\mathrm{Re}\, s > 1$
$0 < \alpha \leq 1$

CASE I: $\alpha = 1$

$$\zeta(s,1) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \zeta(s)$$

CASE II: $\alpha = 1/2$

$$\zeta(s, 1/2) = \frac{1}{(1/2)^s} + \frac{1}{(3/2)^s} + \frac{1}{(5/2)^s} + \frac{1}{(7/2)^s} + \cdots$$

$$= 2^s \left[ 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots \right]$$

$$= 2^s \left[ \zeta(s) - \frac{\zeta(s)}{2^s} \right]$$

$$= (2^s - 1)\, \zeta(s)$$

$0 < a < q$

**CASE III:** $\alpha = a/q$ $\quad (a,q) = 1 \quad a, q \in \mathbb{N}$

$$\zeta(s, a/q) = \sum_{n \geq 0} \frac{1}{(n + a/q)^s} = \frac{1}{(a/q)^s} + \frac{1}{(1 + a/q)^s} + \cdots$$

$$= q^s \sum_{n \geq 0} \frac{1}{(nq + a)^s} = q^s \sum_{\substack{m \geq 1 \\ m \equiv a(q)}} \frac{1}{m^s}$$

$(\mathbb{Z}/q\mathbb{Z})^\times$

$$\mathbb{1}(m \equiv a(q)) = \frac{1}{\phi(q)} \sum_{\chi(q)} \bar{\chi}(a)\, \chi(m)$$

$\chi \to$ DIRICHLET CHARACTER MOD $q$

$$q^s \sum_{\substack{m \geq 1 \\ m \equiv a (q)}} \frac{1}{m^s} = q^s \sum_{m \geq 1} \frac{1}{m^s} \frac{1}{\phi(q)} \sum_{\chi (q)} \bar{\chi}(a) \chi(m)$$

$$= \sum_{\chi (q)} \frac{q^s}{\phi(q)} \bar{\chi}(a) \boxed{\sum_{m \geq 1} \frac{\chi(m)}{m^s}}$$

DIRICHLET

$L(s, \chi)$

$$\zeta(s, a/q) = \frac{q^s}{\phi(q)} \sum_{\chi (q)} \bar{\chi}(a) L(s, \chi)$$

$$\boxed{\zeta(s, a/q) = \frac{q^s}{\phi(q)} \sum_{\chi(q)} \bar{\chi}(a) L(s, \chi)}$$

CASE IV :  $\alpha \notin \Phi$

???

# PROPERTIES OF $\zeta(s,\alpha)$

① ANALYTIC ON $\mathbb{C} \setminus \{1\}$, WITH SIMPLE POLE AT $s=1$ w/ RESIDUE 1.

② NO EULER PRODUCT (EXCEPT $\alpha \in \{1/2, 1\}$).

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (\mathrm{Re}(s) > 1)$$

RIEMANN ZETA

$$\zeta(s,\alpha) = \sum_{n\geq 0} \frac{1}{(n+\alpha)^s}$$

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

③ SATISFIES A "FUNCTIONAL EQUATION"

$$\zeta(1-s,\alpha) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi i s/2}\, \underbrace{P(s,\alpha)} + e^{\frac{\pi i s}{2}}\, P(s,-\alpha)\right)$$

$$P(s,\alpha) = \sum_{n\geq 1} \frac{e^{2\pi i n\alpha}}{n^s}$$

$\alpha = 1$

$$P(s,1) = \zeta(s)$$

RIEMANN : $\zeta(1-s) = \chi(s)\,\zeta(s)$

$$\chi(s) = \frac{\Gamma(s)\cos \pi s/2}{2^{s-1}\pi^s}$$

$(\alpha \neq 1, \tfrac{1}{2})$

$$\zeta(s,\alpha) = \sum_{n \geq 0} \frac{1}{(n+\alpha)^s}$$

(4) ZERO-FREE FOR $\mathrm{Re}(s) \geq 1+\alpha$

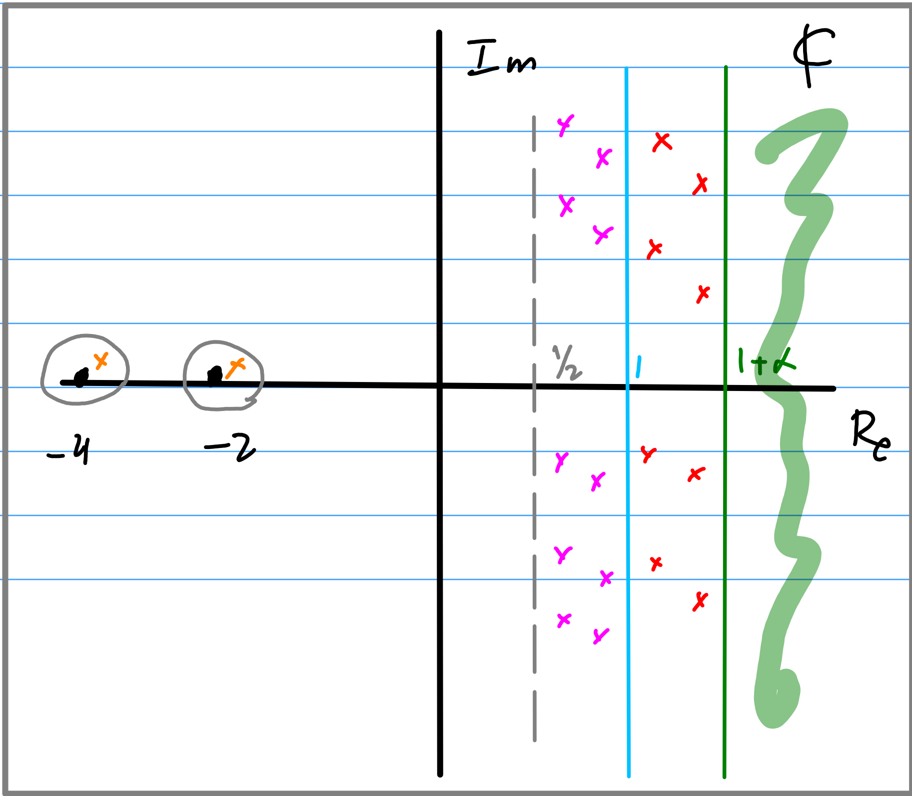
(5) "TRIVIAL" ZEROES NEXT TO − EVEN

(6) DAVENPORT & HEILBRONN (1936) $(\alpha \in \mathbb{Q}$ OR $\alpha \notin \overline{\mathbb{Q}})$

CASSELS (1961) $(\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ ALG. IRR.$)$

(7) GONEK (1979) $(\alpha \in \mathbb{Q}$ OR $\alpha \notin \overline{\mathbb{Q}})$

(8) MINE (2023+) //



## §2 MOMENTS OF $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$

$$M_k(T) = \int_T^{2T} \left| \zeta(\tfrac{1}{2} + it) \right|^{2k} dt$$

MOTIVATION: LINDELÖF HYPOTHESIS IS

$$\zeta(\tfrac{1}{2} + it) \ll_\epsilon t^\epsilon \quad \Longleftrightarrow \quad M_k(T) \ll_{k,\epsilon} T^{1+\epsilon}$$

$$(t \to \infty)$$

($\ll$ = $\leq$ UPTO CONSTANT)

* HARDY - LITTLEWOOD (1916) $M_1(T) \sim T \log T$
* INGHAM (1927) $M_2(T) \sim \frac{1}{2\pi^2} T (\log T)^4$

(...)

* SOUND + HARPER + VARIOUS SUBSETS OF {HEAP, RADZIWILL, SOUND} FOR $k > 0$

$$T(\log T)^{k^2} \ll M_k(T) \ll T(\log T)^{k^2}$$

UNCONDITIONAL

CONDITIONAL ON RH

$0 \leq k \leq 2 \rightarrow$ UNCONDITIONAL

CONJ. (FOLKLORE)

$$M_k(T) = \int_T^{2T} |\zeta(1/2 + it)|^{2k}\, dt \sim c_k \, T (\log T)^{k^2}$$

WHY? SELBERG C.L.T.

$t \in [T, 2T]$

$$\log |\zeta(1/2 + it)| \approx N\left(0, \tfrac{1}{2} \log\log T\right)$$

$$|\zeta|^{2k} = \exp\big(2k \underbrace{\log|\zeta|}_{X}\big) \qquad \mathbb{E}[\exp(2kX)]$$

CONJECTURES:

* CONREY - GHOSH (1980s) : $c_k = \dfrac{a_k g_k}{k^2!}$ ARITHMETIC (✓) GEOMETRIC (??)

$(g_1 = 1,\ g_2 = 2)$

* CONREY - GHOSH (1998) : $g_3 = 42$

* CONREY - GONEK (2001) : $g_4 = 24024$

* KEATING - SNAITH (2000) : $g_k = k^2! \prod_{j=1}^{k-1} \dfrac{j!}{(k+j)!}$

(USES ANALOGY W/ RANDOM MATRICES)

# GONEK-HUGHES-KEATING (2007)

"HYBRID EULER-HADAMARD FORMULA"

EULER HADAMARD

$$\zeta(s) \approx \underbrace{P_X(s)}_{\text{USES PRIMES } p \le X} \; \underbrace{Z_X(s)}_{\text{USES ZEROES } \frac{1}{\log X} \text{ AWAY FROM } s}$$

**Thm** :

$$\frac{1}{T}\int_T^{2T} \left|P_X\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \sim a_k \left(e^{\gamma}\log X\right)^{k^2}$$

**Conj** :

$$\frac{1}{T}\int_T^{2T} \left|Z_X\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \sim \frac{g_k}{k^2!}\left(\frac{\log T}{e^{\gamma}\log X}\right)^{k^2}$$

$\downarrow$

USING RMT

$T \longrightarrow \infty$

## SPLITTING CONJECTURE:

$$\frac{1}{T}\int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \sim \left(\frac{1}{T}\int_T^{2T} \left|P_X\left(\tfrac{1}{2}+it\right)\right|^{2k} dt\right) \times \left(\frac{1}{T}\int_T^{2T} \left|Z_X\left(\tfrac{1}{2}+it\right)\right|^{2k} dt\right)$$

$$a_k \cancel{\left(e^{\gamma} \log X\right)^{k^2}}$$

$$\frac{g_k}{k^2!}\left(\frac{\log T}{\cancel{e^{\gamma} \log X}}\right)^{k^2}$$

§3 MOMENTS OF $\zeta(s, a/q)$

$$M_k(T;\alpha) = \int_T^{2T} |\zeta(1/2+it, \alpha)|^{2k} \, dt$$

* RANE (1980):

$M_1(T,\alpha) \sim T \lg T$ FOR ALL $0 < \alpha \leq 1$.

NO DIOPH. CONDITIONS

RECALL,

$$\zeta(s, a/q) = \frac{q^s}{\phi(q)\,\chi(q)} \sum \bar{\chi}(a)\, L(s,\chi)$$

$$\zeta^k = \left( \sum \text{PRODUCTS OF } L\text{-FUNCTIONS} \right)$$

$$|\zeta|^{2k} = \zeta^k \bar{\zeta}^k = \sum \text{PRODUCTS OF } L\text{-FUNCTIONS}$$

$$\mathcal{L}^{\ell} = \prod L(s,\chi)^{\ell_\chi}$$

AFTER INTEGRATING $\displaystyle\int_T^{2T} \left|\mathcal{L}^{\ell}\left(\tfrac{1}{2}+it\right)\right|^2 dt$

CONJ (S. 2023, PUBLISHED IN MATH. PROC. CAMBRIDGE PHIL. SOC.)

IF $\alpha = a/q \in \mathbb{Q}$,

$$M_k(T;\alpha) \sim C_k(\alpha)\, T (\log T)^{k^2}$$

WHERE

$$C_k(\alpha)/C_k = \frac{q^k}{\varphi(q)^{2k-1}} \prod_{p \mid q} \left[ \sum_{m=0}^{\infty} \binom{m+k-1}{k-1}^2 p^{-m} \right]^{-1}$$

$C_k$ FROM $\zeta(s)$

NOTE: INDEPENDENT OF $a$!

Thm (S. 2023*, PUBLISHED IN MATH. PROC. CAMBRIDGE PHIL. SOC.)

$$M_2(T;\alpha) = \int_T^{2T} \left|\zeta(1/2+it,\alpha)\right|^4 dt$$

$$\sim \frac{1}{2\pi^2}\left[\prod_{p|q}\left(1-\frac{1}{p+1}\right)\right] T (\log T)^4$$

COMPARE

$$\int_T^{2T} |\zeta(1/2+it)|^4 dt \sim \frac{1}{2\pi^2} T(\log T)^4$$

* : INDEPENDENTLY, ANDERSSON (2006)

## §4 MOMENTS OF $\zeta(s,\alpha)$ $(\alpha \notin \mathbb{Q})$

CONJ. (HEAP–S. 2025, PUBLISHED IN CRELLE'S JOURNAL)

FOR ALMOST ALL* $\alpha$

$$M_k(T;\alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2}+it,\alpha)|^{2k}\,dt \sim k!\;T(\log T)^k$$

* : LEBESGUE MEASURE

(INCLUDING ALGEBRAIC IRRATIONAL $\alpha$ WITH DEG $\geq k$)

<u>Thm</u> (HEAP – S., 2025)

IF $\mu(\alpha) < 3$ & $\alpha \notin \mathbb{Q}$ & $0 \le k \le 2$,

$$M_k(T;\alpha) = \int_T^{2T} |\zeta(1/2 + it, \alpha)|^{2k}\, dt \asymp T(\log T)^{k}.$$

$$\boxed{\text{CORRECT ORDER OF MAGNITUDE!}}$$

FOLLOWS FROM

WHY?

**Thm** ( HEAP – S. , 2025 )

IF $\boxed{\mu(\alpha) < 3}$ & $\alpha \notin \mathbb{Q}$

$$M_2(T;\alpha) = \int_T^{2T} \left|\zeta(1/2+it,\alpha)\right|^4 dt \ll T(\log T)^2 .$$

$$\boxed{\text{COMPARE : } \quad M_2(T; a/q) \asymp_q T(\log T)^4} \to \text{WHY?}$$

WHY:

① 2 LOG'S INSTEAD OF 4 LOG'S ?

$$\zeta(s,\alpha) = \sum_{n\geq 0} \frac{1}{(n+\alpha)^s}$$

AFE:

$$\zeta\left(\frac{1}{2}+it,\alpha\right) \approx \sum_{n\leq T^{1/2}} \frac{1}{(n+\alpha)^{1/2+it}}$$

$$|\zeta|^4 = \zeta\cdot\zeta\cdot\bar{\zeta}\cdot\bar{\zeta} = \sum_{n_1,n_2,n_3,n_4\leq T^{1/2}} \frac{1}{\left[\prod (n_j+\alpha)\right]^{1/2}} \left[\frac{(n_3+\alpha)(n_4+\alpha)}{(n_1+\alpha)(n_2+\alpha)}\right]^{it}$$

$$\int_T^{2T} |y|^4 \, dt = \sum_{n_1, n_2, n_3, n_4 \le T^{1/2}} \frac{1}{\left[\prod (n_j + \alpha)\right]^{1/2}} \left[\frac{(n_3+\alpha)(n_4+\alpha)}{(n_1+\alpha)(n_2+\alpha)}\right]^{it}$$

$$\int_T^{2T} \left[\frac{(n_3+\alpha)(n_4+\alpha)}{(n_1+\alpha)(n_2+\alpha)}\right]^{it} dt$$

$$\alpha = \frac{1}{(\log T)^4}$$

M.T. $\rightarrow$ $(n_1+\alpha)(n_2+\alpha) = (n_3+\alpha)(n_4+\alpha)$

$\alpha \notin \mathbb{Q} \Rightarrow \{n_1, n_2\} = \{n_3, n_4\}$ $\quad (\log T)^2$

WHY :

(2) $\mu(\alpha) < 3$ (DIOPHANTINE CONDITION ON RATIONAL APPROXIMATIONS OF $\alpha$)

$$(n_1+\alpha)(n_2+\alpha) \not\approx (n_3+\alpha)(n_4+\alpha)$$

$$\alpha \approx \frac{n_1+n_2-n_3-n_4}{n_3 n_4 - n_1 n_2}$$