

VC DIMENSION IN GROUPS

BASED ON JOINT
WORKS W/

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§1 DEFINITION

(G, \cdot) : FINITE GROUP OF ORDER N .

$$S \subseteq G$$

S SHATTERS $Y = \{y_1, \dots, y_k\}$

$$\forall \emptyset \neq A \subseteq Y, \exists x \in G \text{ s.t. } xS \cap Y = A$$

THE VAPNIK-CHERVOXENKIS DIMENSION OF $S \subseteq G$
($VC\text{-dim}_G(S)$) IS THE SIZE OF THE LARGEST
SHATTERED BY S .

N.B. : $VC\text{-dim}_G(S) \leq \log_2 N.$ ($N := |G|$)

e.g. $G = \mathbb{Z}/11\mathbb{Z}$ $S = \{1^2, 2^2, 3^2, 4^2, 5^2\}$
 $= \{1, 3, 4, 5, 9\}$

e.g.

$$G = \mathbb{Z}/11\mathbb{Z}$$

$$11 < 2^4$$

$$Y = \{0, 1, 2\}$$

$$Y = \{0, 1, 2, 3\}$$

$$S = \{1^2, 2^2, 3^2, 4^2, 5^2\}$$

$$= \{1, 3, 4, 5, 9\}$$

$$S-1 = \{0, 2, 3, 4, 8\}$$

$$\text{VCdim}_G(S) = 3$$

$$0, \boxed{1}, \underline{2}, \boxed{3}, \boxed{4}, \boxed{5}, 6, 7, 8, \boxed{9}, 10$$

$$(S-2) \cap Y = \{0, 3\}$$

NOTE:

$$\mathcal{F}(S) = \{ xS : x \in G \}$$

SET-SYSTEMS

$$\mathcal{H}(S) = \{ \mathbb{1}_{xS} : x \in G \}$$

CLASSIFIERS

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

§2 MAIN QUESTION

Q. $G = (\mathbb{Z}/N\mathbb{Z}, +)$ (N PRIME)
 $\mathcal{A} = \{x^2 : x \in \mathbb{Z}/N\mathbb{Z}\}$

WHAT IS VC-dim $G(\mathcal{A})$ AS $N \rightarrow \infty$?

Thm (MCDONALD-S. - WYMAN, 2025 IN DISCRETE MATH.)

LET $G = (\mathbb{Z}/N\mathbb{Z}, +)$ & $\mathcal{S} = \{x^2 : x \in (\mathbb{Z}/N\mathbb{Z})\}$.

THEN, AS $N \rightarrow \infty$ THROUGH THE PRIMES,

$$\text{VC-dim}_G(\mathcal{S}) \geq \left(\frac{1}{2} - o(1)\right) \log_2 N$$

NOTE: GENERALIZES TO $G = (\mathbb{F}_q, +)$ &

$$\mathcal{S}_\tau = \{x^\tau : x \in \mathbb{F}_q\} \quad \left(\begin{array}{l} q \text{ PRIME POWER} \\ \tau \mid q-1 \end{array} \right)$$

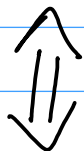
SKETCH OF PROOF

χ : LEGENDRE SYMBOL MOD N .

$$\chi(n) = \begin{cases} 1 & n = \square \\ -1 & n \neq \square \\ 0 & n = 0 \end{cases}$$

$$\frac{1}{2} \chi \approx \frac{1+\chi}{2}$$

S SHATTERS $Y = \{y_1, \dots, y_k\}$



$\forall 0 \leq A \subseteq Y, \exists x \in \mathbb{Z}/N\mathbb{Z}$ s.t. $(x+d) \cap Y = A$

FOR SIMPLICITY TAKE $A=Y$.

NEED TO : $\exists x$ s.t. $Y - x \subseteq d$ $Y = \{y_1, \dots, y_k\}$
SHOW

$$\sum_x \prod_{j=1}^k \mathbb{1}_d(y_j - x) > 0$$

$$\sum_x \prod_{j=1}^k \mathbb{1}_{\mathcal{A}}(\gamma_k - x)$$

$$\mathbb{1} \approx \frac{1+x}{2}$$

$$\sum_x \left[\frac{1}{2^k} + \dots \right]$$

> 0

$$\left[\sum \chi(\dots) \right]$$

SMALL

$$\sum_{x=0}^{N-1} \chi(f(x))$$

WEIL'S BOUND FOR CHARACTER.

CONJ. (MCDONALD - S. - WYMAN , CORRECTED FORM BY
RODGERS - S.)

FOR $G = (\mathbb{Z}/N\mathbb{Z}, +)$ & $\mathcal{A}_r = \{x^r : x \in \mathbb{Z}/N\mathbb{Z}\}$

$$VC_{\text{dir } G}(\mathcal{A}_r) = (1 + o(1)) \log_r N$$

AS $N \rightarrow \infty$ ALONG $N \equiv 1 \pmod{r}$. N PRIME.

$$0 < p < 1$$

§3

RANDOM MODEL

(G, \cdot) \longrightarrow

NOT NECESSARILY ABELIAN.

S \longrightarrow

RANDOMLY SELECTED

$$S \sim \text{BERN}(G, p)$$

FIX $0 < p < 1$,

$$\Pr [x \in S] = p \quad (\text{i.i.d. FOR } x \in G)$$

\Leftrightarrow

$$\Pr [S = W] = p^{|W|} (1-p)^{N-|W|}$$

$W \rightarrow$ DETERMINISTIC

HEURISTIC : SQUARES ARE PSEUDORANDOM

$$\mathcal{A} \approx \text{BERN}(\mathbb{Z}/N\mathbb{Z}, 1/2)$$

MORE GENERALLY

$$\mathcal{A}_r \approx \text{BERN}(\mathbb{Z}/N\mathbb{Z}, 1/r).$$

Thm: (RODGERS - S., 2025+)

FIX $0 < p < 1$. SET $1/r = \text{MIN}(p, 1-p)$

LET (G, \cdot) BE ANY FINITE GROUP OF ORDER N . LET $S \subseteq G$ BE A BERNOULLI SAMPLED RANDOM SUBSET WITH PROB p .

THEN ASYMPTOTICALLY ALMOST SURELY,
AS $N \rightarrow \infty$,

$$\text{VC dim}_G(S) = (1 + o(1)) \log_r N.$$

PROB
OF
FAILURE
 $O(N^{-\eta})$