MA 26500 (FALL 2023, SEC: 704)

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SECTION WEB PAGE https://www.math.purdue.edu/~sahay5/fall2023/ma26500/index.html

NOTE: ALL IMAGES ARE FROM THE TEXTBOOK (LAY, LAY, MCDONALD, 6th ed.)

ANNOUNCEMENTS/NOTES

1. NO CLASS ON WEDNESDAY. FRIDAY CLASS ALSO ON ZOOM.

2. NO OFFICE HOURS OR H.W. THIS WEEK

   ¡ PLEASE WORK ON H.W. NOW ANYWAY !

3. RECORDING OF CLASS WILL BE UPLOADED.

4. TRY TO KEEP VIDEOS ON.

5.1 EIGENVECTORS & EIGENVALUES

EXAMPLE 1 Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix} \rightarrow \text{NOT} \propto \vec{u}.$$

?? $$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\vec{v}$$

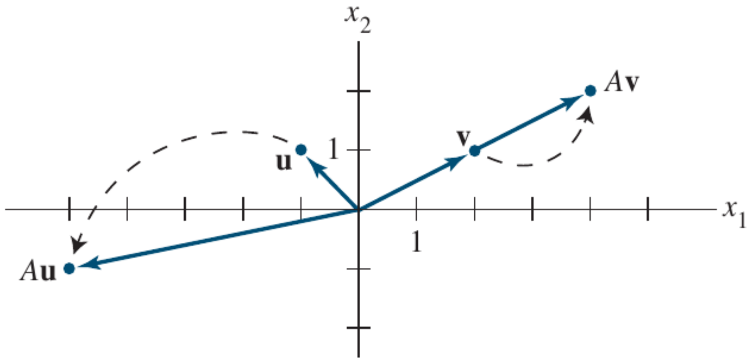

FIGURE 1 Effects of multiplication by A.

MATRIX ONLY SCALES EIGENVECTOR

→ NO CHANGE IN DIRECTION

NOT ALLOWING $\vec{x} = \vec{0}$!

DEFINITION An **eigenvector** of an $n \times n$ matrix A is a $\boxed{\text{nonzero}}$ vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

<u>NOTES</u> : ① NON-TRIVIAL?

② EIGENVALUES <u>CAN</u> BE ZERO

e.g. $A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\vec{v}$

$\vec{v}$ IS AN EIGENVECTOR w/ EIGEN VALUE λ.

EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are **u** and **v** eigenvectors of A?

$$A\vec{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

NOT AN E.V.

$\vec{u}$ IS AN EIGENVECTOR OF A WITH EIGENVALUE -4.

EXAMPLE 3 Show that 7 is an eigenvalue of matrix A in Example 2, and find the corresponding eigenvectors.

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

$\lambda = 7$ IS AN EIGENVALUE.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A\vec{x} = 7\vec{x} \Rightarrow \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 7\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + 6x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 7x_1 \\ 7x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -6x_1 + 6x_2 \\ -2x_1 + 2x_2 \end{bmatrix} = 0$$

$3 \times$ (EQN 2)

$$-6x_1 + 6x_2 = 0$$

$$-2x_1 + 2x_2 = 0$$

$$\boxed{-2x_1 + 2x_2 = 0}$$

$(1, 1)$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ IS EIGENVECTOR FOR $\lambda = 7$.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

GIVEN (A, λ)

$$A\vec{x} = \lambda x$$

$$\Rightarrow \quad A\vec{x} = (\lambda I)\vec{x}$$

$$\Rightarrow \quad (A - \lambda I)\vec{x} = \vec{0}$$

SYSTEM TO SOLVE $\rightarrow$ $\left[\, A - \lambda I \;\middle|\; \vec{0} \,\right]$

NEED ONLY ONE NON-TRIVIAL SOLN.

OBSERVE: IF $(\vec{v}, \lambda)$ IS AN EIGENPAIR, THEN SO IS $(k\vec{v}, \lambda)$ FOR ANY NON-ZERO SCALAR k.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \end{bmatrix} = 7 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$A(k\vec{v}) = k[A\vec{v}] = k[\lambda\vec{v}] = \lambda[k\vec{v}]$$

$\{k\vec{v} : k \in \mathbb{R}\} \longrightarrow \text{Span}\{\vec{v}\} \rightarrow$ SUBSPACE OF $\mathbb{R}^n$

IN FACT, FOR AN $n \times n$ MATRIX WITH EIGENVALUE λ, THE SET OF EIGENVECTORS FORMS A SUBSPACE

$\hookrightarrow$ EIGENSPACE OF A, (CORRESPONDING TO λ)

$(A, \lambda) \to$ FIXED

$\Updownarrow$

ALL SOLNS. OF $A\vec{x} = \lambda\vec{x}$

$\Updownarrow$

ALL SOLNS. OF $(A - \lambda I)\vec{x} = \vec{0}$ $\Leftrightarrow$ $\vec{x} \in \mathrm{NUL}(A - \lambda I)$

∴ AFTER FINDING AN EIGENVALUE, THE EIGENSPACE IS A ROW-REDUCTION PROBLEM!

EXAMPLE 4 Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

$\lambda = 2$ $\qquad$ $(A - 2I)\vec{x} = \vec{0}$

$$\left[A - \lambda I \mid \vec{0} \right]$$

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$2x_1 - x_2 + 6x_3 = 0$$

$$\Rightarrow \quad x_1 = \frac{1}{2}x_2 - 3x_3$$

BOUND : x_1

FREE : x_2, x_3

$$\vec{x} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

EIGENSPACE OF $\lambda = 2$ = NULLSPACE OF $A - 2I$ = $\text{Span}\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

e.v. e.v.

WHAT DOES IT MEAN IF λ IS AN EIGENVALUE?

$\Updownarrow$

$\exists \vec{x}$ s.t. $A\vec{x} = \lambda\vec{x}$

$\Rightarrow \underbrace{(A-\lambda I)}_{B}\vec{x} = \vec{0}$ HAS A NON-TRIVIAL SOLN

Q. WHEN DOES $B\vec{x} = \vec{0}$ HAVE A NON-TRIVIAL SOLN.?

A. $\det B = 0 \Leftrightarrow \text{RANK}(B) < n$.

$\therefore$ IF λ IS AN EIGENVALUE $\Rightarrow \det(A-\lambda I) = 0$

DETOUR :

§ 5.2 THE CHARACTERISTIC EQUATION

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

EXAMPLE 1 Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) - 3 \times 3 = \lambda^2 + 4\lambda - 21$$

$\therefore$ IF λ IS AN EIGENVALUE,

$$\lambda^2 + 4\lambda - 21 = 0$$

$$(\lambda + 7)(\lambda - 3) = 0$$

$$\therefore \lambda = -7 \quad \text{OR} \quad \lambda = 3.$$

RECALL:

THEOREM 3

Properties of Determinants

Let A and B be $n \times n$ matrices.

a. A is invertible if and only if $\det A \neq 0$.

b. $\det AB = (\det A)(\det B)$.

c. $\det A^T = \det A$.

d. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A.

e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

EXAMPLE 3 Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (5-\lambda)^2 (3-\lambda)(1-\lambda)$$

$\Rightarrow \quad \lambda = 5$ (TWICE), $\lambda = 3$, $\lambda = 1$

FACT: IF A IS TRIANGULAR, THEN THE DIAGONAL ENTRIES ARE THE EIGENVALUES.

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$$

WARNING!! → NOT TRUE IF NON-TRIANGULAR.

($A \to n \times n$ MATRIX)

FACT: $\det(A - \lambda I)$ IS A POLYNOMIAL OF DEGREE n.

→ CHARACTERISTIC POLYNOMIAL → $p(\lambda) = \det(A - \lambda I)$

→ (ALGEBRAIC) MULTIPLICITY.

↳ IF λ IS A ZERO OF $\det(A - \lambda I)$ w/ MULTIPLICITY.

REMARK: # OF EIGENVALUES w/ MULTIPLICITY $\leq n$.