

MA 26500 (FALL 2023, SEC: 704)

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SECTION

WEB PAGE

<https://www.math.purdue.edu/~sahay5/fall2023/ma26500/index.html>

NOTE: ALL IMAGES ARE
FROM THE TEXTBOOK
(LAY, LAY, MCDONALD, 6th ed.)

ANNOUNCEMENTS / NOTES

1. IN-PERSON CLASSES RESTART FROM NEXT WEEK.
(LET TRA OFFICE HOURS ON WEP.)

2. NO OFFICE HOURS OR H.W. THIS WEEK

! PLEASE WORK ON H.W. NOW ANYWAY!

3. H.W. 22 & 23 ARE DUE ON TUESDAY (10/24)
AT 11:00 PM.

4. RECORDING OF CLASS WILL BE UPLOADED.

5. TRY TO KEEP VIDEOS ON.

RECALL :

$A \rightarrow n \times n$ MATRIX

Defn : THE MATRIX A HAS EIGENVALUE $\lambda \in \mathbb{R}$
CORRESPONDING TO EIGENVECTOR $\vec{x} \in \mathbb{R}^n$
 $(\vec{x} \neq \vec{0})$



$$A\vec{x} = \lambda\vec{x}$$

H.B : (λ, \vec{x}) IS ALSO CALLED AN EIGENPAIR
(OF A)

RECALL :

λ IS AN EIGENVALUE OF A

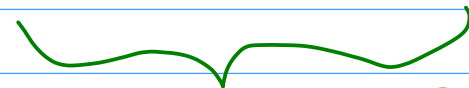


$$\det (A - \lambda I) = 0 \quad \} \rightarrow \text{CHARACTERISTIC EQUATION.}$$

RECALL: GIVEN AN EIGENVALUE λ OF A ,

SET OF EIGENVECTORS = NULLSPACE $(A - \lambda I)$
 $A\vec{x} = \lambda\vec{x}$ $(A - \lambda I)\vec{x} = 0$


EIGENSPACE
OF λ


CAN BE FOUND
BY ROW-REDUCTION
(GAUSSIAN ELIMINATION)

$$A\vec{x} = \lambda\vec{x} \Leftrightarrow (A - \lambda I)\vec{x} = 0$$

($A \rightarrow n \times n$ MATRIX)

FACT: $\det(A - \lambda I)$ IS A POLYNOMIAL OF DEGREE n IN λ .

\rightarrow CHARACTERISTIC POLYNOMIAL $\rightarrow P_A(\lambda) = \det(A - \lambda I)$

\rightarrow (ALGEBRAIC) MULTIPLICITY.

\hookrightarrow IF λ IS A ZERO OF $\det(A - \lambda I)$ W/ MULTIPLICITY.

REMARK: # OF EIGENVALUES W/ MULTIPLICITY $\leq n$.

EXAMPLE 4 The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$.
Find the eigenvalues and their multiplicities.

$$\det(A - \lambda I)$$

$$\lambda^4 [\lambda^2 - 4\lambda - 12]$$

$$\det(A - \lambda I) = \lambda^4 (\lambda - 6) (\lambda + 2)$$

$\lambda = 6$
w/ MULT. 4

$\lambda = 6$
w/ MULT. 1

$\lambda = -2$
w/ MULT. 1

ALGORITHM FOR EIGENPROBLEM

1. COMPUTE CHAR. POLY. ($\det(A - \lambda I)$)
2. FIND ITS ROOTS $\lambda_1, \dots, \lambda_k$ (w / MULTIPLICITY)
3. FOR EACH λ_j , SOLVE THE EQUATION
 $(A - \lambda_j I) \vec{v} = 0$ TO GET EIGENVECTORS.
(EIGENS PACES)

THEOREM I

The eigenvalues of a triangular matrix are the entries on its main diagonal.

$$A = \begin{bmatrix} a_{11} & * & * \\ 0 & a_{22} & * \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & * & * \\ 0 & a_{22} - \lambda & * \\ 0 & 0 & a_{33} - \lambda \end{vmatrix} = \underbrace{(a_{11} - \lambda)}_{a_{11}} \underbrace{(a_{22} - \lambda)}_{a_{22}} \underbrace{(a_{33} - \lambda)}_{a_{33}}$$

EXAMPLE 5 Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$.

E.V. = $\{3, 0, 2\}$

$\{4, 1, 4\}$

e.g. $A, \lambda = 3$

$$(A - 3I) \vec{x} = 0$$

SOLNS
IN \vec{x} .

$$\begin{bmatrix} 0 & 6 & -8 \\ 0 & -3 & 6 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{E_{20s}} \begin{bmatrix} 0 & 6 & -8 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 6 & -8 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$



$$x_3 = 0$$

$$x_2 = \frac{8}{6} x_3 = 0$$

$x_1 \rightarrow$ FREE.

EIGENSPACE
OF $(A, 3)$

→ $\{ (x_1, 0, 0) : x_1 \in \mathbb{R} \}$

$$\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

THEOREM 2

If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A , then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

Pf $n=2$

$$\begin{array}{cc} \lambda_1 & , & \lambda_2 \\ \downarrow & & \downarrow \\ \mathbf{v}_1 & & \mathbf{v}_2 \end{array}$$

SUPPOSE \mathbf{v}_1 & \mathbf{v}_2 ARE L.D. \times

$$\vec{\mathbf{v}}_1 = k \vec{\mathbf{v}}_2$$

$$\lambda_1 \vec{\mathbf{v}}_1 = A \vec{\mathbf{v}}_1 = A(k \vec{\mathbf{v}}_2) = k (A \vec{\mathbf{v}}_2) = k \lambda_2 \mathbf{v}_2$$

$$\vec{\mathbf{v}}_1 = k \vec{\mathbf{v}}_2, \quad \boxed{\lambda_1 \vec{\mathbf{v}}_1} = k \lambda_2 \mathbf{v}_2 = \boxed{\lambda_2 \vec{\mathbf{v}}_1}$$

$$\lambda_1 \vec{v}_1 = \lambda_2 \vec{v}_1 \Rightarrow (\lambda_1 - \lambda_2) \vec{v}_1 = \vec{0}$$

$$\lambda_1 = \lambda_2 \quad \zeta = \lambda_1 - \lambda_2 = 0$$

$$\vec{v}_1 = \vec{0}$$

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if

r. The number 0 is *not* an eigenvalue of A .

$$0 \text{ IS AN E.V.} \Rightarrow \exists \vec{x} \neq 0, \quad A\vec{x} = \vec{0}$$

$$\Rightarrow A\vec{x} = \vec{0}$$

$\Rightarrow A$ IS NOT INVERTIBLE.

$$0 \text{ IS NOT AN EIGENVALUE} \Rightarrow \det(A - \lambda I) \text{ DOESN'T HAVE } \lambda=0 \text{ AS A ROOT}$$

$$\Rightarrow \det A \neq 0$$

$\Rightarrow A$ IS INVERTIBLE

SIMILARITY

$A, B \rightarrow n \times n$ MATRICES

Defn : A IS SIMILAR TO B

$$A = PBP^{-1} \quad \left(\exists \text{ INVERTIBLE MATRIX } P. \right)$$

$$\rightarrow P^T AP = \cancel{P^{-1} P} B \cancel{P P^{-1}} = B$$

(OR EQUIVALENTLY $B = P^{-1}AP$)

$$B = QAQ^{-1}$$

NOTE : SYMMETRY

$$A \approx_{\text{SIM}} B \Leftrightarrow B \approx_{\text{SIM}} A$$

THEOREM 4

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

$$A \sim_{\text{SIM}} B \implies \det(A - \lambda I) = \det(B - \lambda I)$$

$$\xrightarrow{\text{Pf}} \det(A - \lambda I) = \det(PBP^{-1} - \lambda I)$$

RECALL

$$\det A^{-1} = \frac{1}{\det A}$$

$$PI P^{-1} = I$$

$$\det(P \boxed{B} P^{-1} - P \lambda I P^{-1})$$

$$\begin{aligned} &= \det \left[P (B - \lambda I) P^{-1} \right] = (\det P) \cdot \det(B - \lambda I) (\det P^{-1}) \\ &= \det(B - \lambda I) \cdot (\cancel{\det P}) (\cancel{\det P^{-1}}) \end{aligned}$$

WARNINGS :

①

SAME EIGENVALUES



SIMILARITY

e.g.

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



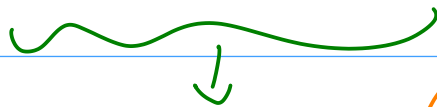
$$P(2I)P^{-1} = 2P \cdot P^{-1} = 2I$$

②

SIMILARITY



ROW EQUIVALENCE.



EROs DON'T PRESERVE
EIGENVALUES

(2x GENERAL)