MA 26500 (FALL 2023, SEC: 704)

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SECTION WEBPAGE https://www.math.purdue.edu/~sahay5/fall2023/ma26500/index.html

NOTE : ALL IMAGES ARE FROM THE TEXTBOOK (LAY, LAY, MCDONALD, 6th ed.)

ANNOUNCEMENTS / NOTES

1. IN-PERSON CLASSES RESTART FROM NEXT WEEK.
   (EXTRA OFFICE HOURS ON WED.)
2. NO OFFICE HOURS OR H.W. THIS WEEK

   ¡ PLEASE WORK ON H.W. NOW ANYWAY !
3. H.W. 22 & 23 ARE DUE ON TUESDAY (10/24) AT 11:00 PM.
4. RECORDING OF CLASS WILL BE UPLOADED.
5. TRY TO KEEP VIDEOS ON.

RECALL: $A \to n \times n$ MATRIX

Defn: THE MATRIX A HAS EIGENVALUE $\lambda \in \mathbb{R}$ CORRESPONDING TO EIGENVECTOR $\vec{x} \in \mathbb{R}^n$

$(\vec{x} \neq \vec{0})$

$$\Updownarrow$$

$$A\vec{x} = \lambda\vec{x}$$

N.B: $(\lambda, \vec{x})$ IS ALSO CALLED AN EIGENPAIR (OF A)

RECALL :

λ IS AN EIGENVALUE OF A

$$\Updownarrow$$

$$\det(A - \lambda I) = 0$$ → CHARACTERISTIC EQUATION.

RECALL: GIVEN AN EIGENVALUE λ OF A,

SET OF EIGENVECTORS = NULLSPACE $(A-\lambda I)$

$$\underbrace{A\vec{x}=\lambda\vec{x}}_{\text{EIGENSPACE OF } \lambda} \qquad \underbrace{(A-\lambda I)\vec{x}=0}_{\text{CAN BE FOUND BY ROW-REDUCTION (GAUSSIAN ELIMINATION)}}$$

$$A\vec{x}=\lambda\vec{x} \iff (A-\lambda I)\vec{x}=0$$

($A \to n \times n$ MATRIX)

FACT: $\det(A - \lambda I)$ IS A POLYNOMIAL OF DEGREE n IN λ.

→ CHARACTERISTIC POLYNOMIAL → $P_A(\lambda) = \det(A - \lambda I)$

→ (ALGEBRAIC) MULTIPLICITY.

↳ IF λ IS A ZERO OF $\det(A - \lambda I)$ w/ MULTIPLICITY.

REMARK: # OF EIGENVALUES w/ MULTIPLICITY $\leq n$.

EXAMPLE 4 The characteristic polynomial of a 6×6 matrix is $\underbrace{\lambda^6 - 4\lambda^5 - 12\lambda^4}_{\det(A - \lambda I)}$. Find the eigenvalues and their multiplicities.

$$\lambda^4 \left[\lambda^2 - 4\lambda - 12 \right]$$

$$\det(A - \lambda I) = \underbrace{\lambda^4}\underbrace{(\lambda - 6)}\underbrace{(\lambda + 2)}$$

$\lambda = 0$ w/ MULT. 4

$\lambda = 6$ w/ MULT. 1.

$\lambda = -2$ w/ MULT. 1.

ALGORITHM FOR EIGENPROBLEM

1. COMPUTE CHAR. POLY. $(\det(A - \lambda I))$
2. FIND ITS ROOTS $\lambda_1, \ldots, \lambda_k$ (W/ MULTIPLICITY)
3. FOR EACH λ_j, SOLVE THE EQUATION

$$(A - \lambda_j I)\vec{v} = 0$$

TO GET EIGENVECTORS.

(EIGENSPACES)

THEOREM 1 The eigenvalues of a triangular matrix are the entries on its main diagonal.

$$A = \begin{bmatrix} a_{11} & * & * \\ 0 & a_{22} & * \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & * & * \\ 0 & a_{22}-\lambda & * \\ 0 & 0 & a_{33}-\lambda \end{vmatrix} = \underbrace{(a_{11}-\lambda)}_{a_{11}}\underbrace{(a_{22}-\lambda)}_{a_{22}}\underbrace{(a_{33}-\lambda)}_{a_{33}}$$

EXAMPLE 5 Let $\underbrace{A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}}_{\text{E.V.} = \{3,0,2\}}$ and $\underbrace{B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}}_{\{4,1,4\}}$.

e.g. A, $\lambda = 3$

$$(A - 3I)\vec{x} = 0 \longrightarrow \begin{bmatrix} 0 & 6 & -8 \\ 0 & -3 & 6 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} 0 & 6 & -8 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

SOLNS IN $\vec{x}$.

$$\begin{bmatrix} 0 & 6 & -8 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \quad x_3 = 0$$

$$x_2 = \frac{8}{6} x_3 = 0$$

$x_1 \to$ FREE.

EIGENSPACE OF $(A, 3)$ $\longrightarrow$ $\{ (x_1, 0, 0) : x_1 \in \mathbb{R} \}$

$$\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

THEOREM 2 If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\boxed{\{\mathbf{v}_1, \dots, \mathbf{v}_r\}}$ is linearly independent.

Pf $n = 2$

$\lambda_1 \; , \; \lambda_2$

$\downarrow \quad \downarrow$

$v_1 \quad v_2 .$

SUPPOSE v_1 & v_2 ARE L.D. ✗

$$\vec{v}_1 = k \vec{v}_2$$

$$\lambda_1 \vec{v}_1 = A\vec{v}_1 = A(k\vec{v}_2) = k(A\vec{v}_2) = k\lambda_2 v_2$$

$$\vec{v}_1 = k\vec{v}_2 , \quad \boxed{\lambda_1 \vec{v}_1} = k\lambda_2 v_2 = \boxed{\lambda_2 \vec{v}_1}$$

$$\lambda_1 \vec{v}_1 = \lambda_2 \vec{v}_1 \Rightarrow (\lambda_1 - \lambda_2)\vec{v}_1 = \vec{0}$$

$\lambda_1 = \lambda_2 \Leftarrow \lambda_1 - \lambda_2 = 0$

$\vec{v}_1 = \vec{0}$

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if

r. The number 0 is *not* an eigenvalue of A.

0 IS AN E.V. $\Rightarrow \exists \vec{x} \neq 0$, $A\vec{x} = 0\vec{x}$

$\Rightarrow A\vec{x} = \vec{0}$

$\Rightarrow$ A IS NOT INVERTIBLE.

0 IS NOT AN EIGENVALUE $\Rightarrow$ $\det(A - \lambda I)$ DOESN'T HAVE $\lambda = 0$ AS A ROOT

$\Rightarrow \det A \neq 0$

$\Rightarrow$ A IS INVERTIBLE

SIMILARITY

$A, B \to n \times n$ MATRICES

Defn : A IS SIMILAR TO B

$\Updownarrow$

$A = PBP^{-1}$ ($\exists$ INVERTIBLE MATRIX P.)

$P^{-1} A P = \cancel{P^{-1} P} B \cancel{P^{-1} P} = B$

(OR EQUIVALENTLY $B = P^{-1} A P$)

$B = Q A Q^{-1}$

NOTE : SYMMETRY $\quad A \simeq_{SIM} B \iff B \simeq_{SIM} A$

THEOREM 4 If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

$$A \simeq_{SIM} B \implies \det(A - \lambda I) = \det(B - \lambda I)$$

Pf

$$\det(A - \lambda I) = \det(PBP^{-1} - \lambda I)$$

$$PIP^{-1} = I$$

RECALL

$$\det A^{-1} = \frac{1}{\det A}$$

$$\det(P\boxed{BP^{-1}} - P\lambda I P^{-1})$$

$$= \det[P(B - \lambda I)P^{-1}] = (\det P) \cdot \det(B - \lambda I)(\det P^{-1})$$

$$= \det(B - \lambda I) \cdot \cancel{(\det P)}\cancel{(\det P^{-1})}$$

WARNINGS : ① SAME EIGENVALUES $\not\Rightarrow$ SIMILARITY

e.g. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}$

$P(2I)P^{-1} = 2P \cdot P^{-1} = 2I$

② SIMILARITY $\not\Leftrightarrow$ $\underbrace{\text{ROW EQUIVALENCE.}}$

EROs DON'T PRESERVE (IN GENERAL) EIGENVALUES