MA 30300 (FALL 2025)

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SECTION
WEB PAGE https://www.math.purdue.edu/~sahay5/fall2025/ma30300/index.html

NOTE : ALL IMAGES ARE FROM THE TEXTBOOK
(EDWARDS, PENNEY, CALVIS 6th ed.)

ANNOUNCEMENTS / NOTES

1. CLASSES SHOULD BE BACK IN-PERSON ON FRIDAY
   (STAY TUNED FOR ANNOUNCEMENT)

2. MIDTERM 1 GRADING IS DELAYED.

3. O12 : TODAY, O13 : FRIDAY, H5 : SATURDAY, O14 : MONDAY
   H6 : TUESDAY (NON-STANDARD!)

4. RECORDING OF CLASS WILL BE UPLOADED TO KALTURA.

5. TRY TO KEEP VIDEOS ON.

REVIEW: EULER'S METHOD

THE GOAL IS TO SOLVE FIRST ORDER IVPs

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

USING NUMERICAL METHODS.

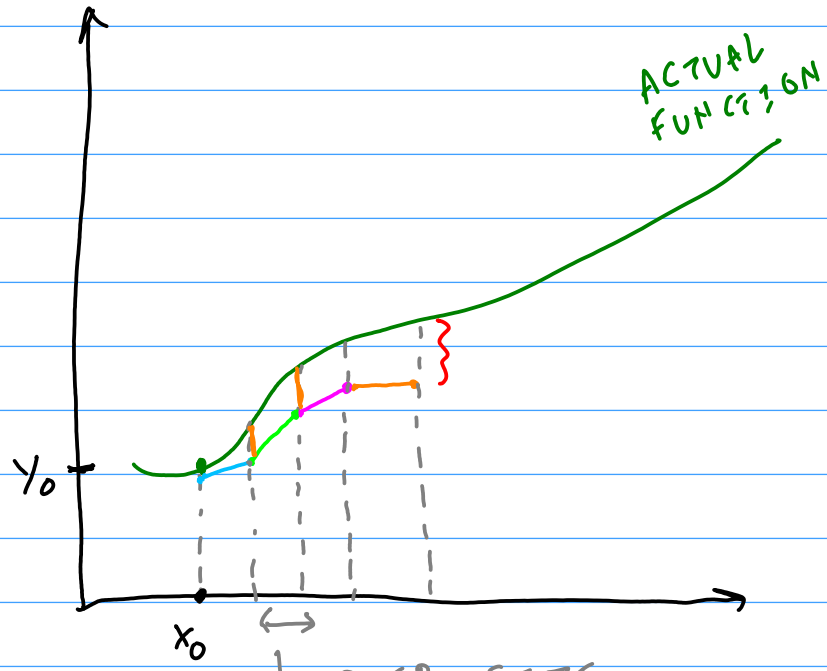
KEY IDEA: APPROXIMATE BY POLYGONAL CURVE, WITH EACH SEGMENT $\approx$ TANGENT LINE.

POLYGONAL $\rightarrow$ PIECE-WISE LINEAR

SOURCES OF ERROR:

(a) LINE SEGMENT

(b) $f(x_j, y_j) \approx f(x_j, y(x_j))$



SCHEMATIC OF EULER'S METHOD

TYPES OF ERROR:

LOCAL

GLOBAL

(x_1, y_1) $f(x_1, y_1)$

(x_2, y_2) $f(x_2, y_2)$

ALGORITHM (EULER)

INPUT: IVP $dy/dx = f(x, y)$ $y_0 = y(x_0)$

STEP SIZE h

OF STEPS n (OR TARGET $x_n = x_0 + nh$)

OUTPUT: A TABLE OF VALUE (x_j, y_j) FOR $0 \leq j \leq n$ WITH $y_j \simeq y(x_j)$.

UPDATE: FOR $1 \leq j \leq n$,

$x_j = x_{j-1} + h$, $y_j = y_{j-1} + h f(x_{j-1}, y_{j-1})$

e.g. $\frac{dy}{dx} = x + y$, $\boxed{y(0) = 1}$ $\quad y(x_0) = y_0$

$h = 0.1$ $\qquad n = 2$

j	x_j	y_j	$f(x_j, y_j)$
0	0	1	1
1	0.1	1.1	1.2
2	0.2	1.22	—

$f(x_0, y_0) = 0 + 1 = 1$

$x_1 = x_0 + h = 0.1$

$y_1 = y_0 + h\, f(x_0, y_0)$

$= 1 + (0.1)(1)$

$= 1.1$

$f(x_1, y_1) = 0.1 + 1.1 = 1.2$

$y_2 = y_1 + h\, f(x_1, y_1)$

$= 1.1 + (0.1)(1.2) = 1.1 + 0.12$

x	Euler Method, $h = 0.1$ Values of y
0.1	1.1000
0.2	1.2200
0.3	1.3620
0.4	1.5282
0.5	1.7210
0.6	1.9431
0.7	2.1974
0.8	2.4872
0.9	2.8159
1.0	3.1875

BACKWARD EULER

KEY IDEA: USE THE OTHER (RIGHT) ENDPOINT TO COMPUTE SLOPE OF POLYGONAL SEGMENTS.

EULER: $\dfrac{y_j - y_{j-1}}{x_j - x_{j-1}} \approx f(x_{j-1}, y_{j-1})$ (LEFT)

BACKWARD EULER: $\dfrac{y_j - y_{j-1}}{x_j - x_{j-1}} \approx f(x_j, y_j)$ (RIGHT)

IMPLICIT UPDATE $x_j = x_{j-1} + h$

ALGORITHM (BACKWARD EULER)

INPUT: IVP $\left(dy/dx = f(x,y) \quad y_0 = y(x_0) \right)$

STEP SIZE h

OF STEPS n (OR TARGET $x_n = x_0 + nh$)

OUTPUT: A TABLE OF VALUE (x_j, y_j) FOR $0 \le j \le n$ WITH $y_j \simeq y(x_j)$.

UPDATE: FOR $1 \le j \le n$,

$x_j = x_{j-1} + h$,

$$\boxed{y_j = y_{j-1} + h f(x_j, y_j)}$$

IMPLICIT

$f \rightarrow$ SIMPLE

$$f(x,y) = e^{x+y}$$

$$y_j = y_{j-1} + h\, e^{x_j + y_j}$$

!!

(NUMERICAL)

ISSUES

(a) Thm: ERROR IN EULER'S ALGORITHM IS $\propto h$.

THIS IS VERY SLOW

(b) BACKWARD EULER REQUIRES NUMERICALLY SOLVING IMPLICIT UPDATE STEP.

SOLUTION? MIX THE TWO ALGORITHMS!

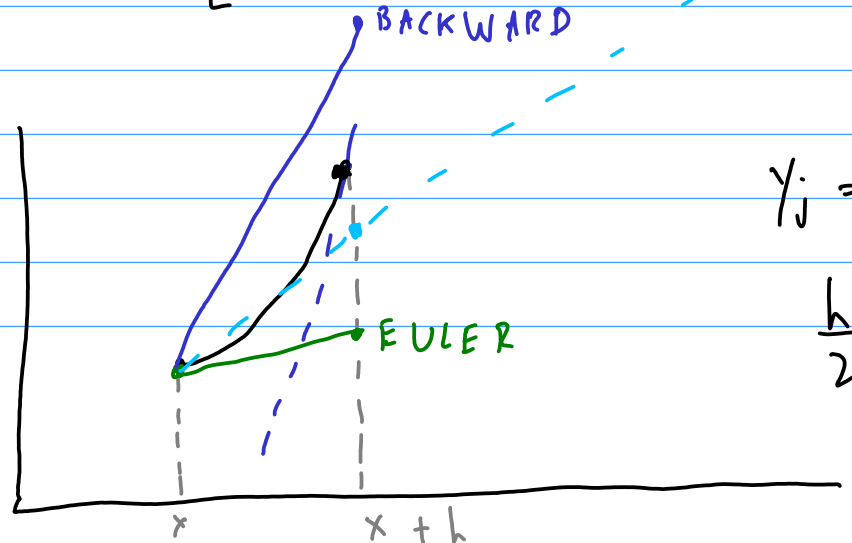
IMPROVED EULER

KEY IDEA : USE AVERAGE OF SLOPES AT ENDPOINTS

$$\frac{y_j - y_{j-1}}{x_j - x_{j-1}} \approx \frac{1}{2}\left[f(x_{j-1}, y_{j-1}) + f(x_j, y_j) \right]$$

IMPLICIT

y_j APPEARS ON BOTH SIDES



$$y_j = y_{j-1} + \frac{h}{2}\left[f(x_{j-1}, y_{j-1}) + f(x_j, y_j) \right]$$

IMPROVED EULER

FURTHER IDEA: USE EULER FOR A FIRST APPROXIMATION u_j

PREDICTOR $u_j = y_{j-1} + h f(x_{j-1}, y_{j-1})$

CORRECTOR $y_j = y_{j-1} + \frac{h}{2}\left[f(x_{j-1}, y_{j-1}) + f(x_j, u_j) \right]$

$u_j \rightarrow$ FIRST GUESS

$y_j \rightarrow$ SECOND GUESS / CORRECTED

ALGORITHM (IMPROVED EULER)

INPUT: IVP $\left(dy/dx = f(x,y) \qquad y_0 = y(x_0) \right)$

STEP SIZE h

OF STEPS n (OR TARGET $x_n = x_0 + nh$)

OUTPUT: A TABLE OF VALUE (x_j, y_j) FOR $0 \le j \le n$ WITH $y_j \simeq y(x_j)$.

UPDATE: FOR $1 \le j \le n$,

$$x_j = x_{j-1} + h, \qquad u_j = y_{j-1} + h f(x_{j-1}, y_{j-1})$$

$$y_j = y_{j-1} + \frac{h}{2}\left[f(x_{j-1}, y_{j-1}) + f(x_j, u_j) \right]$$

$u_j \rightarrow$ FIRST GUESS FOR $y(x_j)$

EXAMPLE OF PREDICTOR-CORRECTOR ALGORITHM

Thm: ERROR IN IMPROVED EULER IS $\propto h^2$

e.g.

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

$h = 0.1$ $\quad$ $n = 2$

$f(x,y) = x + y$ $\quad y(0) = 1$ $\quad h = 0.1$

j	x_j	u_j	Y_j	$f(x_j, u_j)$	$f(x_j, Y_j)$
0	0	–	1	–	1
1	0.1	1.1	1.11	1.2	1.21
2	0.2	1.231	1.24205	1.431	–

$$Y_2 = Y_1 + \frac{h}{2}\left[f(x_1, Y_1) + f(x_2, u_2) \right] = 1.11 + \frac{0.1}{2}\left[1.21 + 1.431 \right]$$

CALC :

$$u_1 = y_0 + f(x_0, y_0) = 1 + (0.1)(1) = 1.1$$

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, u_1) \right]$$

$$= 1 + \frac{0.1}{2}\left[1 + 1.2 \right] = 1 + (0.1)(1.1)$$

$$= 1.11$$

$$u_2 = y_1 + hf(x_1, y_1) = (1.11) + (0.1)(1.21) = 1.11 + 0.121$$

$$= 1.231$$

SOLVE THIS EXACTLY

Figure 2.5.4

Euler and improved Euler approximations to the solution of $dy/dx = x + y, y(0) = 1$.

x	Euler Method, $h = 0.1$ Values of y	Euler Method, $h = 0.005$ Values of y	Improved Euler, $h = 0.1$ Values of y	Actual y
0.1	1.1000	1.1098	1.1100	1.1103
0.2	1.2200	1.2416	1.2421	1.2428
0.3	1.3620	1.3977	1.3985	1.3997
0.4	1.5282	1.5807	1.5818	1.5836
0.5	1.7210	1.7933	1.7949	1.7974
0.6	1.9431	2.0388	2.0409	2.0442
0.7	2.1974	2.3205	2.3231	2.3275
0.8	2.4872	2.6422	2.6456	2.6511
0.9	2.8159	3.0082	3.0124	3.0192
1.0	3.1875	3.4230	3.4282	3.4366

Figure 2.5.5

Improved Euler approximation to the solution of **Eq. (9)** with step size $h = 0.005$.

x	Improved Euler, Approximate y	Actual y
0.0	1.00000	1.00000
0.1	1.11034	1.11034
0.2	1.24280	1.24281
0.3	1.39971	1.39972
0.4	1.58364	1.58365
0.5	1.79744	1.79744
0.6	2.04423	2.04424
0.7	2.32749	2.32751
0.8	2.65107	2.65108
0.9	3.01919	3.01921
1.0	3.43654	3.43656

IMPROVED EULER IS DUE TO HEUN.
(c.f. WIKIPEDIA)

OBSERVATION:

UPDATE EQUATION $y_j = y_{j-1} + h\,k$

$k = k_1 = f(x_{j-1}, y_{j-1})$ (EULER)

$k = \tilde{k}_2 = f(x_j, y_j)$ (BACKWARD EULER)

$k = \frac{k_1 + k_2}{2}$, $k_2 = f(x_j, u_j) \approx f(x_j, y_j) = \tilde{k}_2$ (IMPROVED EULER)

NEXT TIME: RUNGE-KUTTE

k → WEIGHTED AVERAGE OF 4 k's!