MA 303 (§ 764), SAMPLE MIDTERM 1

WE FINISH HERE THE DISCUSSION FROM CLASS.

IN CLASS, WE SHOWED THAT THE COEFFICIENT MATRIX

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

HAS EIGENVALUES -1 (WITH MULT. 2) & 2 (WITH MULT. 1).

FURTHER, WE COMPUTED EIGENSPACES

$\lambda = -1$ HAS EIGENVECTOR $\vec{v}_1 = (2, 1, -1)$

$\lambda = 2$ HAS EIGENVECTOR $\vec{v}_3 = (1, 2, 1)$

ANSATZ : (λ, v) EIGENPAIR GIVES SOL N

$$e^{\lambda t} \vec{v}$$

$\therefore$ WE HAVE TWO L.I. SOLNs. $e^{-t}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ & $e^{2t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

IT REMAINS TO FIND $\vec{v}_2$, A RANK 2 GENERALIZED E.VECTOR OF $\lambda = -1$, WHICH EXISTS SINCE $\lambda = -1$ HAS DEFECT $2 - 1 = 1$.

THERE ARE TWO WAYS TO DO THIS.

METHOD 1 : SOLVE $(A+I)\vec{v}_2 = \vec{v}_1$ IN $\vec{v}_2$

THIS IS EQUIVALENT TO ROW REDUCING

$$[A+I \mid \vec{v}_1] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

BY A SHORT CALCULATION, WE FIND THE RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

IF $v_3 = (c_1, c_2, c_3)$ THEN WE CAN CHOOSE c_3 (SAY $= 0$). WE FIND $c_1 = 2$ & $c_2 = -1$

THUS $\vec{v}_2 = (2, -1, 0)$ [CHECK !]

$\therefore$ ANSATZ : $e^{\lambda t}\left[t\vec{v}_1 + \vec{v}_2 \right]$ WHERE $(A - \lambda I)\vec{v}_2 = \vec{v}_1 \neq 0$

& $(A - \lambda I)\vec{v}_1 = 0$

$\therefore$ ONE MORE SOLN : $e^{-t}\begin{bmatrix} 2t+2 \\ t-1 \\ -t \end{bmatrix}$

GENERAL SOLN. $= c_1 e^{-t}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2t+2 \\ t-1 \\ -t \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

METHOD 2 : SINCE $(A+I)^2 \vec{v}_2 = 0$

(BY DEFN. OF RANK 2 G.E.V.)

FIND $(A+I)^2 = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 1 & 2 & 4 \end{bmatrix}$

CLEARLY ROW-EQUIVALENT TO $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

HENCE, $\text{NULL}(A+I) = \{(-2u-4v, u, v)\}$

($u, v \in \mathbb{R}$ ARE PARAMETERS)

SOME CHOICES OF u & v WILL GIVE RANK-1 E.VECTORS (e.g. $u=-1, v=1$ GIVE $\vec{v}_1$ FROM ABOVE).

MOST CHOICES WILL GIVE RANK-2 G.E.Vs. (e.g. $u=-1, v=0$ GIVES $\vec{v}_2$ FROM PREVIOUS METHOD)

CRUCIAL POINT: THE G.E.V. OBTAINED THIS WAY (LET'S SAY $\tilde{v}_2$) NEED NOT SATISFY

$(A+I)\tilde{v}_2 = v_1$!!

HOWEVER IT WILL BE THE CASE THAT $(A+I)\tilde{v}_2 = \tilde{v}_1$ WHERE $\tilde{v}_1$ IS A DIFFERENT E.VECTOR THAN $\vec{v}_1$

e.g.: TAKING $u = 0$, $v = 1$ GIVES

$$\tilde{v}_2 = (-4, 0, 1)$$

$$(A + I)\tilde{v}_2 = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \neq \vec{v}_1$$

$\therefore$ IN THIS CASE THE SOLN. WOULD BE

$$e^{\lambda t}\left[t\tilde{v}_1 + \tilde{v}_2\right] = e^{-t}\begin{bmatrix} -2t - 4 \\ -t \\ t + 1 \end{bmatrix}$$