MA 30300: Differential Equations And Partial Differential Equations For Engineering And The Sciences

Sample Midterm 2

November 5th, 2025

- This is a sample exam for §764 of MA 303. The actual exam will have fewer problems (this one has an extra problem for practice).
- The exam will be 60 minutes long.
- There are 17 pages.
- Unless stated otherwise, **show all your work and provide justifications.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You may use theorems proved in class provided you accurately state them before using them.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- When the time is over, put down your writing instruments immediately.
- Violating these instructions will be considered an act of academic dishonesty. Any act of dishonesty will be reported to the Office of the Dean of Students. Penalties for such behaviour can be severe and may include an automatic F in the course.

NAME: ______________________________

PUID: ______________________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	15	
2	5	
3	6	
4	6	
5	8	
6	6	
TOTAL	46	

Table of Laplace Transforms and other formulae

Function $f(t)$	Laplace Transform $\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n for $n = 1, 2, 3 \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt) = \frac{e^{kt}-e^{-kt}}{2}$	$\frac{k}{s^2-k^2}$
$\cosh(kt) = \frac{e^{kt}+e^{-kt}}{2}$	$\frac{s}{s^2-k^2}$
$e^{ct}f(t)$	$F(s-c)$
$f'(t)$	$sF(s) - f(0)$
$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$
$e^{at}f(t)$	$F(s-a)$
$-tf(t)$	$F'(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$
$(f * g)(t)$	$F(s)G(s)$
$u(t-a)f(t-a)$	$e^{-as}F(s)$

If $f(t)$ has Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L},$$

the Fourier coefficients are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L}, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L}.$$

The Dirac delta "function" $\delta(t)$ is defined by the property that for any $a \leq b$ and any c,

$$\int_a^b f(t)\delta(t-c)\,dt = \begin{cases} f(c) & \text{if } a \leq c \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

The equations

$$mx'' + kx = \cos \omega_0 t, \qquad my'' + ky = \sin \omega_0 t$$

with $\omega_0 = \sqrt{k/m}$ respectively have resonance solutions

$$x(t) = \frac{1}{2m\omega_0} t \sin \omega_0 t, \qquad y(t) = -\frac{1}{2m\omega_0} t \cos \omega_0 t.$$

The Euler update equations state $y_{j+1} = y_j + hk, \qquad x_{j+1} = x_j + h$, where k is an approximation to the average slope. In the case of the fourth-order Runge-Kutta method,

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$k_1 = f(x_j, y_j), k_2 = f(x_j + \tfrac{1}{2}h, y_j + \tfrac{1}{2}hk_1),$$

$$k_3 = f(x_j + \tfrac{1}{2}h, y_j + \tfrac{1}{2}hk_2), k_4 = f(x_{j+1}, y_j + hk_3).$$

$$mx'' + kx = \cos \omega_0 t, \qquad my'' + ky = \sin \omega_0 t$$

with $\omega_0 = \sqrt{k/m}$ respectively have resonance solutions

$$x(t) = \frac{1}{2m\omega_0} t \sin \omega_0 t, \qquad y(t) = -\frac{1}{2m\omega_0} t \cos \omega_0 t.$$

1. (15 points)

Determine if the following statements are true or false. Fill in your answers in the following table. You do not need to justify your answers.

(a)	The function $f(x)$ which outputs 1 if x is positive, -1 if x is negative and 0 if x is 0 is piece-wise smooth.	
(b)	The update equations of the improved Euler method states $x_{j+1} = x_j + h$ and $y_{j+1} = y_j + f(x_{j+1}, y_{j+1})h$.	
(c)	Any periodic and piece-wise smooth function $f(t)$ can be written as $f(t) = g(t) + h(t)$ where g is odd and h is even.	
(d)	We have that $$\int_{-\pi}^{\pi} \cos mt \cos nt \, dt = 0$$ for all integers $m, n \geq 1$.	
(e)	The improved Euler method is a second-order Runge-Kutta method.	
(f)	The fundamental period of $\sin(2t) + \cos(3t)$ is π.	
(g)	The 1-periodic function which is given by $f(t) = t$ for $-1/2 < t < 1/2$ has $a_0 = 0$.	
(h)	If $f(t)$ is 2π-periodic, then $$a_n = \frac{1}{L} \int_{-\pi/2}^{3\pi/2} f(t) \cos nt.$$	
(i)	The error term in Euler's method is proportional to h where h is the step-size.	
(j)	We have that $$\int_0^1 t^2 \delta(t-2) \, dt = 4.$$	

2. (5 points)

For the following two statements determine whether they are true or false. In either case, provide appropriate justification for your answer (e.g., a counterexample or a brief proof).

(a) The differential equation

$$x'' + 16x = \sum_{n=1}^{\infty} \left[\frac{1 + (-1)^n}{n^2} \right] \cos nt$$

has a resonance solution.

(b) If the function $g(t)$ is 1-periodic and has mean-value M, then the function $f(t) = g(t) - M$ has the Fourier coefficient $a_0 = 0$.

(Continued)

3. (6 points) Consider the initial value problem

$$y' = 2 + xy, \qquad y(0) = -1.$$

Using the improved Euler method, compute an approximation to $y(2)$ with the step size $h = 1$.

(Continued)

4. (6 points) Solve the following initial value problem to obtain $y(t)$:

$$y'' - 3y' + 2y = 5\delta(t-2), \qquad y(0) = y'(0) = 0.$$

Final Answer:

(Continued)

5. (8 points) Find the Fourier series of the 2π-periodic function $f(t)$ given by

$$f(t) = e^{|t|}, \qquad \text{for } -\pi < t < \pi.$$

Final Answer:

(Continued)

6. (6 points) The function $f(t)$ has the Fourier series

$$f(t) = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi t)}{n^2}.$$

Find the solution $x(t)$ with $0 < t < 2$ as a formal Fourier series for the boundary value problem

$$x'' + 3x = f(t), \qquad x(0) = x(2) = 0.$$

Final Answer:

(Continued)

Scratch Work

Scratch Work