PROBLEM SESSION
FRG GRAD SEMINAR

MODERATOR: OFIR GORODETSKY

April 21, 2024

Problem 1. [Anurag Sahay] Carol Wu [11, Thm. 1.1] proved the following: there exist infinitely many primes $p > q$ such that $p - q$ is 47-smooth. Can one improve the constant 47?

Wu's argument works by optimally using the conclusions of Maynard's generalization of the GPY sieve. To improve the constant, probably one will need to modify the sieve itself.

Henryk Iwaniec suggested a generalization: for which sets S of primes and almost primes, can one guarantee that there are infinitely many primes $p > q$ such that $p - q$ belongs to $\langle S \rangle$, the set generated multiplicatively by elements of S?

Problem 2. [Henryk Iwaniec] Establish (conditionally or otherwise) an asymptotic large sieve for sequences $a = b * c$ where b has length Q^2 and c is special and is as long as possible. 'Special' can mean whatever makes the problem tractable, say, 'supported on primes' or 'supported on smooth numbers'.

Problem 3. [Brian Conrey] Given a family $\mathcal{F}$ of L-functions, write $L(f,s)^k = \sum_n \frac{\lambda_{f,k}(n)}{n^s}$. Find a family $\mathcal{F}$ for which one can identify the 2-swap terms in the average

$$\mathop{\mathbb{E}}_{\substack{f \in \mathcal{F} \\ \mathrm{cond}(f) \leqslant Q}} \sum_n \frac{\lambda_{f,20}(n)}{n^{1/2+\alpha}} \phi\left(\frac{\log n}{\log X}\right)$$

where $X = Q^{2+\varepsilon}$ for some $\varepsilon > 0$ and $\mathrm{cond}(f)$ is the analytic conductor of f.

Will Sawin suggested using the recent work [2, 9] on quadratic L-functions in $\mathbb{F}_q[T]$ as a guide.

Vorappan Chandee suggested a large and promising family could be the family of automorphic L-functions considered by Baluyot–Chandee–Li [1].

Problem 4. [Igor Shparlinski] Let $D(N) = (\log N)^{\log 2 + \varepsilon}$. Green and Soundararajan [3] proved that almost all $n \leqslant N$, in natural density sense, are represented by $x^2 + dy^2$ for some $x, y \in \mathbb{Z}$ and $0 \leqslant d \leqslant D(N)$, and this $D(N)$ is optimal. For what function $D(N)$ does this hold for *all* (and not 'almost all') integers?

Henryk Iwaniec suggested addressing either the 'almost all' or the 'all' version of the problem above with d restricted to primes.

Igor Shparlinksi provided the following suggestion by email: "Assume that we have a version of the results of Hooley, Bykovsky, Duke-Friedlander-Iwaniec for square moduli. That is, we know that the roots of the congruences $x^2 = n \bmod y^2)$ are equidistributed, with a power saving, uniformly over n, and for $y \sim Y$. Then there is $c > 0$ such that for some $y \sim Y$ there is $x_0 \ll Y^{2-c}$ and such that $x_0^2 = n \bmod y_0^2$ for some $y_0 \sim Y$. This means $x^2 \ll Y^{4-2c}$. Thus, we can take

Y of order $n^{1/(4-2c)}$ to guarantee that $x_0^2 < n$. Hence for $d = (n - x_0^2)/y_0^2$ we have $n = x_0^2 + dy_0^2$ and $1 \leqslant d \ll n/Y^2 \ll n^{1/(2-c)}$, improving the trivial bound $d \ll n^{1/2}$."

Problem 5. [Henryk Iwaniec] Find a counterexample (or, find a proof) for the following version of the large sieve for prime moduli:

$$\sum_{p \leqslant Q} \sum_{\chi_0 \neq \chi \bmod p} \left| \sum_{n=1}^{N} \chi(n) a_n \right|^2 \ll \left(\frac{Q^2}{\log Q} + N \right) \sum_{n=1}^{N} |a_n|^2. \tag{1}$$

This is known for general a_n supported on primes $n < Q^{1.9999}$.

Igor Shparlinski asked if it possible to establish (1) with some $o(Q^2)$ in place of $Q^2/\log Q$.

Problem 6. [Ofir Gorodetsky] Given $k \geqslant 1$ and a real α, study the variance in short intervals of $d_k(n)e(n\alpha)$ ($e(t) = e^{2\pi i t}$). Namely, establish asymptotics for

$$\frac{1}{X} \int_X^{2X} \left| \sum_{x < n \leqslant x+H} d_k(n) e(n\alpha) - (M(x+H) - M(x)) \right|^2 dx$$

as $X \to \infty$, where $M(x)$ is a suitable approximation for $\sum_{n \leqslant x} d_k(n) e(n\alpha)$. Here $1 \leqslant H \leqslant X$.

For badly approximable α, we expect $M(x) \approx 0$, while for rational numbers $M(x)$ should be taken to be the residue of $F(s)x^s/s$ at $s = 1$ where $F(s) = \sum_n d_k(n) e(n\alpha)/n^s$.

For $\alpha = 0$ this problem was studied by Keating–Rodgers–Roditty-Gershon–Rudnick [6] and others. For $k = 2$ and $\alpha \in \mathbb{Q}$ this is partly explored by Kiuchi and Tanigawa [7]. No results seem to be known for $\alpha \notin \mathbb{Q}$.

Problem 7. [Anurag Sahay] Consider the Hurwitz zeta function defined by $\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s}$. Find $\alpha \notin \mathbb{Q}$ such that

$$\frac{1}{T(\log T)^2} \int_T^{2T} |\zeta(1/2 + it, \alpha)|^4 dt \to \infty$$

holds as $T \to \infty$.

For rational α, the integral grows like $\asymp T \log^4 T$ [10] while for $\alpha \notin \mathbb{Q}$ it grows like $\asymp T \log^2 T$ if the irrationality measure $\mu(\alpha)$ is less than 3 [5].

Problem 8. [Brad Rodgers] Can we find arbitrarily large N a multiplicative function f taking values in $\{-1, 1\}$ (and depending on N) satisfying

$$\sup_t \left| \sum_{n \leqslant N} f(n) e(nt) \right| < 2024\sqrt{N} \log\log N?$$

What is the optimal bound?

Sárközy conjectured that $O(\sqrt{N})$ is not achievable. Montgomery conjectured this bound holds for $N = p - 1$, $f(n) = (n/p)$.

Similar questions might be interesting to investigate when f is allowed to take other values (e.g. $-1, 0, 1$, or roots of unity, or in the unit circle).

Igor Shparlinski asked about other norms (e.g. L^4, L^{10}, etc) than the sup-norm in this problem.

Ofir Gorodetsky asked what happens for the Liouville function. Brad Rodgers commented that the recent work of Hardy [4] discusses the correct random model for this problem.

Will Sawin suggested taking a quadratic character with prime modulus p slightly less than N and modifying it so that it is 1 or -1 at p.

Problem 9. [Ofir Gorodetsky] Fix $k \geqslant 1$ and let $n, m \geqslant 1$. Establish asymptotics for

$$I_{k,1}(N;n,m) = \int_G ([u^n]\det(I-uU)^k)\overline{([u^m]\det(I-uU)^k)}\,dU$$

and

$$I_{k,2}(N;n) = \int_G [u^n](\det(I-uU)^k)\,dU$$

where G is $\mathrm{O}(N)$ or $\mathrm{Sp}(N)$. The motivation is that

$$I_k(N;n) = \int_G \left|[u^n]\det(I-uU)^k\right|^2\,dU$$

was studied for $G = \mathrm{U}(N)$ in [6] by Keating–Rodgers–Roditty-Gershon–Rudnick and for $G = \mathrm{O}(N)$ and $G = \mathrm{Sp}(N)$ in [8] by Kuperberg and Lalin, motivated by questions on the k-fold divisor function. It is easy to see, and is implicit in [6], that $I_{k,1}(N;n,m)$ vanishes if $n \neq m$ and $G = \mathrm{U}(N)$, and that similarly $I_{k,2}(N;n)$ vanishes if $G = \mathrm{U}(N)$. Moreover, $I_k(N,n) = I_{k,1}(N;n,n)$ if $G = \mathrm{U}(N)$. However, for the groups $G = \mathrm{O}(N)$ and $G = \mathrm{Sp}(N)$, there is no *a priori* reason for $I_{k,1}(N;n,m)$ and $I_{k,2}(N;n)$ to vanish and so these integrals might be interesting to study.

References

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