NOTES FOR HAESSIG'S TOPICS IN NT

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Let $(K, \|\cdot\|)$ be a normed field (see [Mon70] for a definition). Since K is a field, we have a canonical map $\mathbb{Z} \to K$ whose kernel is an ideal of the form $m\mathbb{Z}$ where $m = 0$ or $m = p$ is the characteristic of K. Given this canonical map, we will abuse notation and write n for both the integer and the element it maps to in K.

Proposition 1. *If $\zeta \in K$ is a root of unity, then $\|\zeta\| = 1$.*

Proof. First, we prove this for $\zeta = 1$. Note that $1 = 1^2$, and hence, $\|1\| = \|1\|^2$. Now, $1 \neq 0$, so $\|1\| \neq 0$. Thus, by cancelling, we conclude $\|1\| = 1$.

More generally, we have $\zeta^n = 1$ for some $n \in \mathbb{N}$. Thus,

$$\|\zeta\|^n = \|\zeta^n\| = \|1\| = 1$$

and hence $\|\zeta\|$ is a root of unity in $\mathbb{R}_{\geq 0}$. The only such number is 1, and hence $\|\zeta\| = 1$.

□

Definition 2. We say that $(K, \|\cdot\|)$ is non-Archimedean if for every $n \in \mathbb{Z}$,

$$\|n\| \leq 1.$$

Proposition 3. *A normed field is non-Archimedean if and only if it satisfies the ultrametric triangle inequality,*

$$\|x + y\| \leq \max\{\|x\|, \|y\|\},$$

for every $x, y \in K$.

Proof. We first prove the easy direction. Note that if $\zeta \in K$ is a root of unity then $\|\zeta\| = 1$, and further that $\|0\| = 0$. If the ultrametric triangle inequality is true, then by induction, for $n \in \mathbb{N}$,

$$\|n\| \leq \|1\| = 1.$$

Thus, for $n \in \mathbb{Z}$, we get $\|n\| \leq 1$, by recalling that $\|-n\| = \|-1\| \, \|n\| = \|n\|$.

For the other direction, we will make use of the tensor product trick (see [Tao07]).

We want to leverage the fact that norms behave well with products to amplify the usual triangle inequality

$$\|x + y\| \leq \|x\| + \|y\| .$$

Consequently, we will use the fact that $\|x^n\| = \|x\|^n$ extensively.

In particular, we apply the binomial theorem to $(x + y)^n$. Thus, we get that

$$(x + y)^n = \sum_j \binom{n}{j} x^j y^{n-j}$$

Now, $\binom{n}{j} \in \mathbb{Z}$, so $\left\|\binom{n}{j}\right\| \leq 1$. Applying the usual triangle inequality together with this fact on the right of the above equality, we get that

$$\|(x + y)^n\| \leq \sum_{j=0}^{n} \|x\|^j \, \|y\|^{n-j} \leq (n + 1) \max\{\|x\|^n , \|y\|^n\}$$

Now, raising both sides to $1/n$,

$$\|x + y\| \leq (n + 1)^{1/n} \max\{\|x\| , \|y\|\}$$

However, note that on sending $n \to \infty$, $(n + 1)^{1/n} \to 1$, proving the desired inequality.

□

Proposition 4. *If* $\operatorname{char} K = p \neq 0$*, then for all* $n \in \mathbb{Z}$*,* $\|n\| = 1$ *or* $\|n\| = 0$.

Proof. Clearly, we want to show that if $p \nmid n$, then $\|n\| = 1$, as otherwise $n = 0$ in K, giving us that $\|n\| = 0$ as desired.

By Proposition 1, it suffices to show that if $p \nmid n$, then n is a root of unity in K.

However, recall that the image of $\mathbb{Z}$ in K is isomorphic to $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, since $p = \operatorname{char} K$ generates the kernel. However, since $\mathbb{F}_p^\times$ is finite and cyclic, non-zero elements of $\mathbb{F}_p$ are roots of unity, and we are done.

□

This proposition clearly implies that all normed fields of characteristic p are non-Archimedean.

Definition 5. Suppose $(K, \|\cdot\|)$ is non-Archimedean. We define $\mathcal{O}_K$ and $\mathfrak{m}_K$ as follows

$$\mathcal{O}_K = \{x \in K : \|x\| \leq 1\},$$

$$\mathfrak{m}_K = \{x \in K : \|x\| < 1\}.$$

In other words, $\mathcal{O}_K$ is the closed unit ball in K and $\mathfrak{m}_K$ is the open unit ball in K.

Note that the closure of the open unit ball need not be the closed unit ball. Further, it is obvious that $\mathfrak{m}_K \subseteq \mathcal{O}_K$, and that $\mathcal{O}_K$ contains the image of $\mathbb{Z}$ in K.

Proposition 6. *$\mathcal{O}_K$ is a valuation ring for K and $\mathfrak{m}_K$ is the only maximal ideal of $\mathcal{O}_K$. In other words, $\mathcal{O}_K$ is a local integral domain with field of fractions K such that for every $x \in K$ either $x \in \mathcal{O}_K$ or $x^{-1} \in \mathcal{O}_K$.*

Proof. First, we show that $\mathcal{O}_K$ is closed under $+$ and $\cdot$, thereby proving that it is a domain. If $x, y \in \mathcal{O}_K$, then, by the ultrametric Δ-inequality,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \leq 1,$$

and further,

$$\|xy\| = \|x\| \, \|y\| \leq 1,$$

thus showing that $x+y, xy \in \mathcal{O}_K$ as desired.

The proof that $\mathfrak{m}_K$ is closed under $+$ is exactly the same with the last $\leq$ replaced by $<$ in both statements. $\mathfrak{m}_K \subseteq \mathcal{O}_K$ is trivial. To show that $\mathcal{O}_K \mathfrak{m}_K \subseteq \mathfrak{m}_K$, let $x \in \mathcal{O}_K$ and $y \in \mathfrak{m}_K$. Then,

$$\|xy\| \leq \|x\| \, \|y\| \leq \|y\| < 1,$$

giving $xy \in \mathfrak{m}_K$.

Now, suppose $x \in K$ such that $x \notin \mathcal{O}_K$. Thus, $\|x\| > 1$. However, then, $\|x^{-1}\| = \|x\|^{-1} < 1$ and hence $x^{-1} \in \mathcal{O}_K$. This also shows that K is the field of fractions of $\mathcal{O}_K$.

It remains to show that $\mathfrak{m}_K$ is the only maximal ideal in $\mathcal{O}_K$. We use the standard result [AM69, Proposition 1.6] for proving localness. Let $x \in \mathcal{O}_K$ such that $x \notin \mathfrak{m}_K$. Clearly, $\|x\| = 1$, and so, $\|x^{-1}\| = 1$ for $x^{-1} \in K$. In particular, this means that $x^{-1} \in \mathcal{O}_K$, and hence x is a unit in $\mathcal{O}_K$. By an application of the result, $\mathfrak{m}_K$ is the unique maximal ideal of $\mathcal{O}_K$ and we are done.

□

At this point, we want to solve Exercises 1, 5, 7, 9 from [Kob84, pp. 6-7]. 1 is cumbersome, so we skip that. We note that 7 and 9 follow easily from 5. For 7, this is because $\|q\|_p = \|p\|_q 1$ for primes $p \neq q$, while $\|p\|_p, \|q\|_q < 1$; for 9, this is because if $\|n\| \leq 1$ then $\|n\|^\alpha \leq 1$ for every $\alpha > 0$.

Proposition 7. *Let K be a field with two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then, there exists a real number $\alpha > 0$, such that*

$$\|x\|_1 = \|x\|_2^\alpha$$

for every $x \in K$.

Proof. First, we clarify that Koblitz says two norms are equivalent if they have the same Cauchy sequences. In particular, one direction is obvious, since $\|\cdot\|_1 = \|\cdot\|_2^\alpha$ obviously implies that the Cauchy sequences are the same. We will prove the contrapositive of the converse. Namely, we will show that if there is no $\alpha > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ for every $x \in K$, then there is a sequence that is Cauchy in one norm but not Cauchy in the other.

First, suppose that there is an $x \in K$ such that $\|x\|_1 \geq 1$ but $\|x\|_2 < 1$. Since $\|x\|_2 < 1$, $x \neq 1$, and hence $\|1-x\|_1 \neq 0$. Then, observe that $x^n \to 0$ in $\|\cdot\|_2$, and hence, x^n is a Cauchy sequence in $\|\cdot\|_2$. To see that it is not Cauchy in $\|\cdot\|_1$, observe that

$$\left\|x^n - x^{n+1}\right\|_1 \geq \|x\|_1^n \|1-x\|_1 = \|1-x\| > 0,$$

is bounded away from 0 no matter how far we choose n.

By inverting elements and flipping the roles of $\|\cdot\|_1$ and $\|\cdot\|_2$, we can now assume that $\|x\|_1 \sim 1$ is equivalent to $\|x\|_2 \sim 1$ for every $x \in K$ and any relation $\sim\ \in \{<, >, =\}$.

Thus, if we assume there is no α satisfying the hypothesis we want it to, it must be the case that there are $x, y \in K$ such that

$$\|x\|_1 = \|x\|_2^\alpha,$$

$$\|y\|_1 = \|y\|_2^\beta,$$

for $\alpha \neq \beta$, $\alpha, \beta \in \mathbb{R}_{\geq 0}$, and $\|x\|_j \neq 1 \neq \|y\|_j$ for $j = 1, 2$.

By replacing x and y with their inverses if necessary, we can further assume that $\|x\|_1, \|y\|_1 > 1$, and by symmetry we can assume that $\alpha > \beta$.

Now, let $z = x^m y^{-n}$ for positive integers m, n to be fixed later. The choice we will make will show that $\|z\|_2 < 1$ and $\|z\|_1 > 1$, which will give us a witness for the inequivalence of the norms, as discussed above.

In particular, since $\|x\|_2 > 1$, $\log \|x\|_2 > 0$. Thus, $\log \|y\|_2 / \log \|x\|_2$ is a positive real number. Since $\beta < \alpha$, we have that

$$0 < \frac{\beta}{\alpha} \frac{\log \|y\|_2}{\log \|x\|_2} < \frac{\log \|y\|_2}{\log \|x\|_2}.$$

Thus, by density, we can choose a rational number m/n with $m, n > 0$ such that

$$0 < \frac{\beta}{\alpha} \frac{\log \|y\|_2}{\log \|x\|_2} < \frac{m}{n} < \frac{\log \|y\|_2}{\log \|x\|_2}.$$

Rearranging this inequality, we get

$$\log \|z\|_2 = m \log \|x\|_2 - n \log \|y\|_2 < 0,$$

and

$$\log \|z\|_1 = m\alpha \log \|x\|_2 - n\beta \log \|y\|_2 > 0.$$

Applying exp on both inequalities, we are done.

□

Now, suppose $\hat{K}$ is the metric completion of K. Then, due to continuity of the field operations, $\hat{K}$ is actually a field extension of K, and the norm extends by the natural definition $\|\alpha\|_{\hat{K}} = \lim \|\alpha_n\|_K$ where $\alpha_n \in K$, $\alpha_n \to \alpha \in \hat{K}$. We will abuse notation, and refer to both norms by $\|\cdot\|$.

Proposition 8. *Let $(\hat{K}, \|\cdot\|)$ be the completion of $(K, \|\cdot\|)$. Then, $\mathcal{O}_{\hat{K}}$ and $\mathfrak{m}_{\hat{K}}$ are the topological closures respectively of $\mathcal{O}_K$ and $\mathfrak{m}_{\hat{K}}$ under $\|\cdot\|$. Furthermore, $\mathcal{O}_K/\mathfrak{m}_K$ and $\mathcal{O}_{\hat{K}}/\mathfrak{m}_{\hat{K}}$ are isomorphic.*

Proof. In this proof, we denote topological closures by $\bar{\cdot}$. $\mathcal{O}_K \subseteq \mathcal{O}_{\hat{K}}$ is obvious, so it suffices to prove that $\mathcal{O}_{\hat{K}} \subseteq \overline{\mathcal{O}_K}$. Let $x \in \mathcal{O}_{\hat{K}}$ such that $x_n \to x$ for $x_n \in K$. We will show that all but finitely many x_n satisfy $x_n \in \mathcal{O}_K$. Thus, by passing to a subsequence if necessary, we will have shown that x to be a limit of a sequence of elements in $\mathcal{O}_K$, and hence $x \in \mathcal{O}_{\hat{K}}$. To see this, note that, $x_n = x + (x_n - x)$, and hence,

$$\begin{aligned} \|x_n\| &\le \max\{\|x\|, \|x_n - x\|\} \\ &\le \max\{1, \|x_n - x\|\}. \end{aligned}$$

As $n \to \infty$, the rightmost term becomes 1, since $x_n \to x$. Thus, we are done.

Similarly, $\mathfrak{m}_K \subseteq \mathfrak{m}_{\hat{K}}$ is obvious. To see $\mathfrak{m}_{\hat{K}} \subseteq \overline{\mathfrak{m}_K}$, we repeat the above argument to get that for $x \in \mathfrak{m}_{\hat{K}}$, $x_n \to x$, $x_n \in K$,

$$\|x_n\| \le \max\{\|x\|, \|x_n - x\|\}$$

For all large enough n, $\|x_n - x\| < \|x\|$, and since $\|x\| < 1$, we get that $\|x_n\| \le \|x\| < 1$, as desired.

The isomorphism from $\mathcal{O}_K/\mathfrak{m}_K$ to $\mathcal{O}_{\hat{K}}/\mathfrak{m}_{\hat{K}}$ is given by $x + \mathfrak{m}_K \mapsto x + \mathfrak{m}_{\hat{K}}$. Well-definedness follows from $\mathfrak{m}_K \subseteq \mathfrak{m}_{\hat{K}}$, and ring-homomorphismness is trivial. Thus, it suffices to show that this map is surjective. In particular, it suffices to show that if we have $y \in \mathcal{O}_{\hat{K}}$, then there is an $x \in \mathcal{O}_K$ such that $x - y \in \mathfrak{m}_{\hat{K}}$. Thus, $x + \mathfrak{m}_{\hat{K}} = y + \mathfrak{m}_{\hat{K}}$, and hence $y + \mathfrak{m}_{\hat{K}}$ is in the image of hte surjection above.

To do this, recall that we have a sequence $y_n \to y$ such that $y_n \in \mathcal{O}_K$. In particular, picking n large enough so that $\|y_n - y\| < 1$, we can pick $x = y_n$, and hence the inequality implies that $x - y \in \mathfrak{m}_{\hat{K}}$ as claimed. This completes the proof. □

References

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