Partitions and Rademacher's Exact Formula

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Certificate

This is to certify that the project report titled 'Partitions and Rademacher's Exact Formula' submitted by Anurag Sahay and Rijul Saini for their KVPY Summer Project is a record of bonafide work carried out by them under my supervision and guidance.

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About the Project

This document is a record of the topics read by in the Theory of Partitions from [1] and [2], as part of their Summer Project 2012, for the KVPY Fellowship, funded by the Department of Science and Technology (Govt. of India). This 'reading project' was done under the supervision of Prof. Amitabha Tripathi, from 22nd June, 2012 to 22nd July, 2012, and was geared towards first gaining a broad overview of partition theory, and then, eventually, understanding the proof of Rademacher's Exact Formula for the unrestricted partition function $p(n)$. The purpose of this project was to provide some exposure to an advanced field of Number Theory by reading a comparitively more esoteric and involved proof than one would normally encounter in their undergraduate curriculum.

Contents

Chapter 1

Introduction

The Theory of Partitions is a very researched sub-field of Additive Number Theory, and is historically known for some of Hardy and Ramanujan's asymptotic results. The Rademacher formula for the partition function is an astonishing result in Partition Theory, the crowning achievement of the so-called Hardy-Littlewood Circle Method. This report documents some of these famous results.

Any result listed in this report, unless otherwise specified, has been proved either in Chapter 14 of [1] or Chapter 5 of [2]. Furthermore, if the result is listed in this report, we have understood the proof of the result, and grasped at least some of the motivation behind it.

Three theorems in this report have been marked with a star ($\star$). This indicates that the proofs of these propositions is not in the above-mentioned chapters, and furthermore, said proofs are considerably involved, moreso than the rest of the project. While we have skimmed through the proofs of these propositions, and will eventually do them properly, this reading has been deferred until after the project is over. A detailed study of these proofs would involve more time and effort than has already been put into the project. Thus, both these results have been taken for granted for the purposes of this project.

The second chapter of the report focuses on the results contained in Chapter 14 of [1]. The third chapter focuses on the proof of Rademacher's formula given in Chapter 5 of [2], as well as some other ancillary reading from that book required to understand parts of the book.

Remarks are used to point out specific information about theorems, and to provide basic proof outlines where necessary.

Chapter 2

Preliminary Results in Partition Theory

2.1 Introduction

The results enumerated in this chapter are from Chapter 14 of [1], entitled 'Partitions'. The chapter begins by giving some background on the field of Additive Number Theory in general, mentioning and stating some historically influential conjectures and problems, such as the Goldbach Conjecture, the representation of positive integers as sums of squares, and Waring's problem.

The unrestricted partition function $p(n)$ is defined the the following way.

Definition 2.1. The unrestricted partition function $p(n)$ is the number of ways n can be written as a sum of positive integers $\leq n$, that is, the number of solutions of

$$n = a_1 + a_2 + \cdots$$

The number of summands is unrestricted, repetition is allowed and the order of the summands is not taken into account.

The analysis of $p(n)$ forms the majority of the remaining chapter. The chapter begins with the geometric representation of partitions, commonly known as a Ferrers diagram, and uses combinatorial methods to intuitively prove the first theorem of the chapter. The ordinary generating function for partitions is obtained, first with a proof assuming the generating function to be a formal power series, and then by considering the questions of convergence. The celebrated Pentagonal Number Theorem of Euler is then proved, and used immediately to derive Euler's recursion formula for $p(n)$. In the next section, an elementary upper bound for $p(n)$ is obtained and a proof of Jacobi's triple product identity, a generalization of Euler's theorem, is given. The penultimate section deals with a general recurrence formula obtained by the logarithmic differentiation of generating function. In the final section, the remarkable division identites of Ramanujan are listed and a proof is outlined in the exercises.

One of the remarkable asymptotic formulas for $p(n)$, originally by Hardy and Ramanujan is

$$p(n) \sim \frac{e^{K\sqrt{n}}}{4n\sqrt{3}}$$

as $n \to \infty$ where $K = \pi \left(\frac{2}{3}\right)^{\frac{1}{2}}$

This beautiful formula is stated in [1] and an extension is proved an extension of this in [2].

2.2 Major Results

Theorem 2.1. *The number of partitions of n into m parts is equal to the number of partions of n into parts of which the largest is m.*

Remark 2.1. This is a simple and basic application of the Ferrers diagrams; it is obtained by flipping the Ferrers diagram about it's diagonal.

Theorem 2.2 (Euler)**.** *For $|x| < 1$ we have*

$$\prod_{m=1}^{\infty} \frac{1}{1-x^m} = \sum_{n=0}^{\infty} p(n)x^n,$$

where $p(0) = 1$

Definition 2.2. For all $n \in \mathbb{Z}$, the pentagonal numbers $\omega(n)$ is defined by the formula

$$\omega(n) = \sum_{k=0}^{n-1} (3k+1) = \frac{3n(n-1)}{2} + n = \frac{3n^2 - n}{2}$$

Lemma 2.1. *For* $|x| < 1$

$$\prod_{m=1}^{\infty}(1-x^m) = 1 + \sum_{n=1}^{\infty}(p_e(n) - p_o(n))x^n$$

where $p_e(n)$ *and* $p_o(n)$ *are respectively the number of partitions of* n *into even number of unequal parts and odd number of unequal parts.*

Remark 2.2. This can be seen by noting that for any given odd partition, the term corresponding to that on the left is negative, while for any even partition, the term corresponding to that on the left is positive.

Theorem 2.3 (Euler's Pentagonal Number Theorem)**.** *If* $|x| < 1$ *we have*

$$\begin{aligned}\prod_{m=1}^{\infty}(1-x^m) &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots \\ &= 1 + \sum_{n=1}^{\infty}(-1)^n(x^{\omega(n)} + x^{\omega(-n)}) \\ &= \sum_{n=-\infty}^{\infty}(-1)^n x^{\omega(n)}\end{aligned}$$

Remark 2.3. From the previous lemma, this theorem follows by constructing a bijection between the odd partition function and the even partition function for positive integers which are not pentagonal numbers, and showing that the bijection does not work for exactly one partition for the positive integers which are pentagonal numbers.

Theorem 2.4. *Let* $p(0) = 1$ *and define* $p(n) = 0$ *if* $n < 0$*. Then, for* $n \geq 1$ *we have*

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) + \cdots = 0$$

or, equivalently

$$p(n) = \sum_{k=1}^{\infty}(-1)^k\big(p(n-\omega(k)) + p(n-\omega(-k))\big)$$

Remark 2.4. This theorem is an easy corrolary of Euler's Pentagonal Number Theorem by noting the fact that the left hand side of the equality is the generating function of $p(n)$ and using simple generating function methods.

Theorem 2.5. *If $n \geq 1$ we have $p(n) < e^{K\sqrt{n}}$, where $K = \pi\left(\frac{2}{3}\right)^{\frac{1}{2}}$*

Remark 2.5. The author motivates this upper bound by referring to Hardy and Ramanujan's asymptotic relation, and by noting $p(n)x^n < \sum_{k=0}^{\infty} p(n)x^n$, proves the bound using relatively elementary arguments.

Remark 2.6. The author also proves the stronger bound

$$p(n) < \frac{\pi e^{K\sqrt{n}}}{\sqrt{6(n-1)}}$$

for $n > 1$, by a slight change in the base inequality.

Theorem 2.6 (Jacobi's triple product identity)**.** *For complex x and z with $|x| < 1$ and $z \neq 0$ we have*

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^2)(1+x^{2n-1}z^{-2}) = \sum_{m=-\infty}^{\infty} x^{m^2} z^{2m}$$

Theorem 2.7. *For a given set A and a given arithmetical function f, the numbers defined by the equation*

$$\prod_{n \in A}(1-x^n)^{-\frac{f(n)}{n}} = 1 + \sum_{n=1}^{\infty} p_{A,f}(n)x^n$$

satisfy the recursion formula

$$np_{A,f}(n) = \sum_{k=1}^{n} f_A(k)p_{A,f}(n-k)$$

where $p_{A,f}(0) = 1$ and

$$f_A(k) = \sum_{d|k, d \in A} f(d)$$

Remark 2.7. The basic proof relies on an assumption of absolute convergence everywhere and basically involved logarithmic differentiation of the product followed by an interchange of sums.

Remark 2.8. If $f(n) = n$, then $p_{A,f}(n) = p(n)$ and $f_A(k) = \sigma(k)$. Hence,

$$np(n) = \sum_{k=1}^{n} \sigma(k)p(n-k)$$

a remarkable relation connecting a function of multiplicative number theory with one of additive number theory.

2.3 Ramanujan's Partition Identities

The chapter ends with a discussion of Ramanujan's striking divisibility identities, obtained by a detailed examination of MacMohan's table of the partition, which was purportedly obtained by hand.

He states three of Ramanujan's identities, specifically,

$$p(5m+4) \equiv 0 \pmod 5$$

$$p(7m+5) \equiv 0 \pmod 7$$

$$p(11m+6) \equiv 0 \pmod{11}$$

He further mentions two connected identities of Ramanujan:

$$\sum_{m=0}^{\infty} p(5m+4)x^m = 5\frac{\varphi(x^5)^5}{\varphi(x)^6}$$

$$\sum_{m=0}^{\infty} p(7m+5)x^m = 7\frac{\varphi(x^7)^3}{\varphi(x)^4} + 49x\frac{\varphi(x^7)^7}{\varphi(x)^8}$$

where

$$\varphi(x) = \prod_{n=1}^{\infty}(1-x^n)$$

Chapter 3

Rademacher's Exact Formula

3.1 Introduction

Rademacher's formula for the unrestricted partition function, as given in Chapter 5 of [2] is an astonishing formula: looking at it, it is hard to believe it is even an integer, let alone $p(n)$. The formula is given in the following theorem.

Theorem 3.1. *If $n \geq 1$, then $p(n)$ is given by the convergent series*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right)$$

where

$$A_k(n) = \sum_{\substack{0\leq h<k \\ (h,k)=1}} e^{\pi i s(h,k) - 2\pi i n h/k}$$

and $s(h,k)$ is a Dedekind sum, given by

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor\frac{hr}{k}\right\rfloor - \frac{1}{2}\right)$$

The author begins the chapter on a historical note, describing the remarkable form of the asymptotic formulae obtained by Hardy and Ramanujan using the so-called 'Hardy-Littlewood Circle Method', going on to relate how Rademacher's change in the analysis of the problem caused the asymptotic formula to become exact. He gives a plan of the proof in general, and then gets down to the proof itself. The proof of the formula is both hard and non-intuitive, and the presentation does not help motivate the methods used, but merely contains Rademacher's brilliant ideas. The proof uses, in a critical manner, three things: the functional equation of the Dedekind eta function, $\eta(\tau)$, under the group of modular transformations, Farey fractions, and Ford Circles. The Dedekind eta function arises in the problem naturally from it's definition. It is given by

Definition 3.1. If $\mathcal{H} = \{\tau : Im(\tau) > 0\}$ is the upper half plane in the Argand plane, the Dedekind eta function is defined as

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})$$

where $\tau \in \mathcal{H}$

Remark 3.1. From Theorem 2.2, it is clear that

$$F(x) = \prod_{m=1}^{\infty} \frac{1}{1 - x^m} = \frac{e^{i\pi\tau/12}}{\eta(\tau)}$$

where $x = e^{2\pi i\tau}$.

We will document the proof of Rademacher's formula in the following manner: Sections 3.2, 3.3 and 3.4 will detail the background to understanding the proof, Section 3.5 will detail the path of integration and a series of lemmas that are used in the proof and Section 3.6 will detail an outline of the proof assuming the lemmas of Section 3.5.

3.2 Dedekind's Functional Equation

The functional equation satisfied by the Dedekind eta function is given by

Theorem 3.2 ($\star$)**.** *If $ad - bc = 1$ for $a, b, c, d \in \mathbb{Z}, c > 0$ and $\tau \in \mathcal{H}$, we have*

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon(a, b, c, d)[-i(c\tau + d)]^{1/2}\eta(\tau)$$

where

$$\varepsilon = e^{\pi i\left(\frac{a+d}{12c} + s(-d,c)\right)}$$

and

$$s(h, k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor\frac{hr}{k}\right\rfloor - \frac{1}{2}\right)$$

Taking reciprocals, choosing $a = H, c = k, d = -h, b = -\frac{hH+1}{k}$ and $\tau = \frac{iz+h}{k}$ and replacing z by z/k, we get

Theorem 3.3. *Let*

$$x = \exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}\right), \, x' = \exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)$$

where $Re(z) > 0, k > 0, (h,k) = 1,$ *and* $hH \equiv -1 \pmod{k}$. *Then*

$$F(x) = e^{\pi is(h,k)} \left(\frac{z}{k}\right)^{1/2} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) F(x')$$

Remark 3.2. The essential part of this theorem is that if $|z|$ is small, x lies near the root of unity $e^{2\pi ih/k}$ and x' lies near the origin. Hence, aside from a constant factor, $F(x)$ behaves like $z^{1/2}e^{\frac{\pi}{12z}}$. In the proof, $F(x)$ will be replaced by this elementary function, and the resulting error in the answer will be obtained; it will so happen that the error will vanish when a certain quantity will be allowed to go to infinity.

3.3 Farey Fractions

Farey fractions are a common occurence in Number Theory. They are defined in the following manner.

Definition 3.2. The set of Farey fractions of order n denoted F_n, is the set of reduced fractions in the closed interval $[0,1]$ with denominators $\leq n$ listed in increasing order of magnitude.

The author proves some elementary theorems about Farey fractions which are easily obtained and generally known in order to prove one specific theorem, which is the following.

Theorem 3.4. *The set* F_{n+1} *includes* F_n. *Each fraction in* F_{n+1} *which is not in* F_n *is a mediant*[1] *of a pair of consecutive fractions in* F_n. *Moreover, if* $a/b < c/d$ *are consecutive in* F_n, *then they satisfy the unimodular relation* $bc - ad = 1$.

The major use of Farey fractions in the proof comes from this theorem, and the intimate connection between Farey fractions and Ford circles.

[1] The mediant of two reduced fractions a/b and c/d is the fraction $(a+c)/(b+d)$.

3.4 Ford Circles

Like Farey fractions, Ford circles are also a common occurrence in Number Theory, and are intimately connected with the Farey fractions. They are defined in the following manner.

Definition 3.3. Given a rational number h/k with $(h,k) = 1$, the Ford circle belonging to this fraction is denoted by $C(h,k)$ and is that circle in the complex plane with radius $1/(2k^2)$ and center at the point $(h/k) + i/(2k^2)$.

The following theorem makes explicit the relationship between Ford circles and Farey fractions.

Theorem 3.5. *Two Ford circles $C(a,b)$ and $C(c,d)$ are either tangent to each other or they do not intersect. They are tangent if, and only if, $bc-ad = \pm 1$. In particular, Ford circles of consecutive Farey fractions are tangent to each other.*

Remark 3.3. This can be easily seen by using the distance formula between the centers of two Ford circles and comparing it with the sum of their two radii.

The points of tangency between two Ford circles (as given in the following theorem) are required as the construction of Rademacher's path of integration depends crucially on it.

Theorem 3.6. *Let $h_1/k_1 < h/k < h_2/k_2$ be three consecutive Farey fractions. The points of tangency of $C(h,k)$ with $C(h_1,k_1)$ and $C(h_2,k_2)$ are the points*

$$\alpha_1(h,k) = \frac{h}{k} - \frac{k_1}{k(k^2+k_1^2)} + \frac{i}{k^2+k_1^2}$$

and

$$\alpha_2(h,k) = \frac{h}{k} + \frac{k_2}{k(k^2+k_2^2)} + \frac{i}{k^2+k_2^2}$$

Moreover, $\alpha_1(h,k)$ lies on the semicircle whose diameter is the interval $[h_1/k_1, h/k]$

3.5 The path of integration and other results

The path of integration used by Rademacher is ingenious. The basic idea is to construct a path from i to $i+1$ on the imaginary plane by travelling along the Ford circles associated with a sequence of Farey fractions. The explicit constructions is the following: Choose a positive integer N and consider the set of Farey fractions of order N. For each Farey fraction, consider the corresponding Ford circle. The points of tangency of this circle with those circles corresponding to the preceding and succeeding Farey fraction divides the Ford circle into two halves: the lower arc, touching the real axis, and the upper arc. The path, $P(N)$, is the union of all such upper arcs obtained. With this path of integration, we can now state a few easily proven results which will help in estimating the integrals arising in the proof.

Theorem 3.7. *The transformation*

$$z = -ik^2\left(\tau - \frac{h}{k}\right)$$

maps the Ford circle $C(h, k)$ in the τ-plane onto a circle K in the z-plane of radius $1/2$ about the point $z = \frac{1}{2}$ as the center. Further, the point $\alpha_1(h, k)$ and $\alpha_2(h, k)$ are mapped onto the points

$$z_1(h, k) = \frac{k^2}{k^2 + k_1^2} + i\frac{kk_1}{k^2 + k_1^2}$$

and

$$z_2(h, k) = \frac{k^2}{k^2 + k_2^2} - i\frac{kk_2}{k^2 + k_2^2}$$

Theorem 3.8. *For the points z_1 and z_2 we have*

$$|z_1(h, k)| = \frac{k}{\sqrt{k^2 + k_1^2}},\ |z_2(h, k)| = \frac{k}{\sqrt{k^2 + k_2^2}}$$

Moreover, if z is a point on the chord joining z_1 and z_2, we have

$$|z| < \frac{\sqrt{2}k}{N}$$

The length of this chord does not exceed $2\sqrt{2}k/N$

3.6 Outline of the Proof

The starting point behind the proof is a Laurent series, specifically, from Theorem 2.2, we have for each $n \geq 0$, if $0 < |x| < 1$

$$\frac{F(x)}{x^{n+1}} = \sum_{k=0}^{\infty} p(k)x^{k-n-1}$$

This function has a pole at $x = 0$ with residue $p(n)$, hence, by Cauchy's residue theorem we get

$$p(n) = \frac{1}{2\pi i} \oint_C \frac{F(x)}{x^{n+1}} dx$$

where C is any positively oriented simple closed contour which lies inside the unit circle and winds once around the origin. The Hardy-Littlewood circle method chooses a circular contour with radius close to 1. Since the denominator of $F(x)$ vanishes at every root of unity, each of them form a singularity for $F(x)$. This contour is divided into arcs $C_{h,k}$ which lie near the roots of unity $e^{2\pi i h/k}$ for the reduced fraction h/k. Then, after fixing some integer N, the integral can be written as the finite sum,

$$\oint_C = \sum_{\frac{h}{k} \in F_N} \int_{C_{h,k}}$$

As noted in Remark 3.2, on the arcs $C_{h,k}$, $F(x)$ behaves remarkably like an elementary function $\zeta_{h,k}$. To make this notion rigorous, the author does the following steps.

First, the change of variable $x = e^{2\pi i \tau}$. This maps a circle going counter-clockwise with radius $e^{-2\pi}$ centered at the origin in the x-plane to the line segment going from i to $i + 1$ in the τ-plane. This segment is now replaced by Rademacher's path of integration $P(N)$. This gives

$$p(n) = \int_{P(N)} F(e^{2\pi i \tau}) e^{-2\pi i n \tau} d\tau$$

Furthermore, we have

$$\int_{P(N)} = \sum_{\frac{h}{k} \in F_N} \int_{\gamma_{h,k}}$$

where $\gamma_{h,k}$ denotes the upper arc of the Ford circle $C(h, k)$.

Next, the author applies the transformation $z = -ik^2(\tau - h/k)$ and then shows through a series of involved calculations entailing the functional equation that

$$p(n) = \sum_{\frac{h}{k} \in F_N} ik^{-5/2}\omega(h,k)e^{-2\pi inh/k}(I_1(h,k) + I_2(h,k))$$

where

$$\omega(h,k) = e^{\pi is(h,k)}$$

$$I_1(h,k) = \int_{\gamma_{h,k}} \xi_k(z)e^{2n\pi z/k^2}dz$$

$$I_2(h,k) = \int_{\gamma_{h,k}} \xi_k(z)\left\{F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)\right) - 1\right\}e^{2n\pi z/k^2}dz$$

$$\xi_k(z) = z^{1/2}\exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right)$$

Here the $I_1(h,k)$ term is what we get when we replace F by the elementary function. The error term represented by $I_2(h,k)$ must be estimated. It can be shown using the bounds on the chord joining $z_1(h,k)$ and $z_2(h,k)$ that the integrand in $I_2(h,k)$ is $< c|z|^{1/2}$ where c does not depend on z or N. The path in $I_2(h,k)$ can be continuously deformed into the chord joining the two end-points without changing the value of the integral, hence we easily obtain the bound

$$|I_2(h,k)| < Ck^{3/2}N^{-3/2}$$

for some constant C.

Hence, the total error introduced is

$$\begin{aligned}\left|\sum_{\frac{h}{k} \in F_N} ik^{-5/2}\omega(h,k)e^{-2\pi inh/k}I_2(h,k)\right| &< \sum_{\frac{h}{k} \in F_N} Ck^{-1}N^{-3/2} \\ &\leq CN^{-3/2}\sum_{k=1}^{N} 1 \\ &= CN^{-1/2}\end{aligned}$$

Hence the error introduced is $O(N^{-1/2})$.

Furthermore, for the integral $I_1(h,k)$, the author introduces the entire circle K clockwise as the path of integration, and shows in a similar manner that the error introduced is $O(N^{-1/2})$.

Remark 3.4. At this point, it is prudent to point out something not mentioned in the text: the integral along the circle K is an improper integral since the integrand of $I_1(h,k)$ has an essential singularity at the point $z=0$. This becomes more explicit later when a change of variable is introduced.

Here, the author lets $N \to \infty$ to obtain

$$p(n) = i\sum_{k=1}^{\infty} A_k(n)k^{-5/2} \oint_K z^{1/2} \exp\left(\frac{\pi}{12z} + \frac{2\pi z}{k^2}\left(n - \frac{1}{24}\right)\right)dz$$

where

$$A_k(n) = \sum_{\substack{0\le h<k \\ (h,k)=1}} e^{\pi i s(h,k) - 2\pi i n h/k}$$

The author then refers to the following formula in Watson's treatise on Bessel function.

Theorem 3.9 ($\star$).

$$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-\nu-1}e^{t+(z^2/4t)}dt$$

if $c>0$, $Re(\nu)>0$ *and* $I_\nu(z) = i^{-\nu}J_\nu(iz)$.

Now the transformation $w = 1/z$ turns the integral into the form shown in this theorem, with

$$\frac{z}{2} = \left\{\frac{\pi^2}{6k^2}\left(n - \frac{1}{24}\right)\right\}^{1/2}$$

and $\nu = 3/2$.

Remark 3.5. This fact shows that the integral around K was an improper integral in disguise. Further, it gives some insight to why the author chose to map all Ford circles to the circle K. The act of taking the reciprocal is equivalent to the composition of two acts: inversion (in the geometric sense) about the unit circle, followed by conjugation (or reflection about the real axis). It is well-known that inversion maps generalized circles to generalized circles, and hence, that reciprocation would map the circle K to the straight line perpendicular to the real axis passing through $z = 1$ over which the Bessel function's integral is given.

Theorem 3.10 ($\star$)**.** *Bessel functions of half-odd order can be reduced in terms of elementary functions. Specifically,*

$$I_{3/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left(\frac{\sinh z}{z} \right)$$

Putting this all together gives Rademacher's formula:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}} \right)$$

Bibliography

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