Problem Solving Seminar

AMS Grad Student Chapter, University of Rochester

11th November, 2019

Problem 1 ($+++$[*], Putnam 1971)**.** Let c be a real number such that n^c is an integer for every positive integer n. Show that c is a non-negative integer. [1]

Problem 2 (?? Moscow State University (2013) [†])**.** Let $x_1, \cdots, x_k \in S^{n-1} = \mathbb{R}^n$ be points on the unit sphere such that

$$0 \in \texttt{conv}\{x_1, \cdots, x_k\}$$

where conv represents the convex hull. With the convention that $x_{k+1} = x_1$, show that

$$\sum_{j=1}^{k} \|x_j - x_{j+1}\| \geq 4$$

Problem 3 ($++$, [Hat00][‡])**.** For $1 \leq p < \infty$

$$\ell^p(\mathbb{N}) = \left\{ (a_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$

Now, let

$$S_p^{\infty} = \left\{ x \in \ell^p(\mathbb{N}) : \|x\|_{\ell^p(\mathbb{N})} = 1 \right\}$$

Show that S_p^{∞} is contractible.

Further, let

$$V_0 = \{(a_n)_{n=1}^{\infty} : \exists N, \forall n \geq N, a_n = 0\}$$

be the set of sequences that are eventually zero. Show that $V_0 \cap S_p^{\infty}$ is contractible.

Problem 4 (???, Problem 16 from [Jia])**.** Let R be a noncommutative ring with unity, and suppose the elements $x, y \in R$ are such that both $(1 - xy)$ and $(1 - yx)$ are invertible. Show that

$$(1 + x)(1 - yx)^{-1}(1 + y) = (1 + y)(1 - xy)^{-1}(1 + x)$$

[*] $+$ indicates hardness; the more plusses there are, the harder I think the problem is. Conversely, $-$ indicates easiness. ? indicates I don't know the solution, and the number of ?s indicates how hard I think the solution probably is.

[†] Translated by Firdavs from their geometry contest

[‡] This problem was paraphrased by Brian

References

[Hat00] Allen Hatcher. *Algebraic topology.* Cambridge Univ. Press, Cambridge, 2000.

[Jia] Ziling Jiang. Abstract algebra, 18.a24 guest lecture: Fall 2018. Yufei Zhao's webpage. http://yufeizhao.com/a34/fa18/algebra.pdf (version: 2019-11-8).

Hints

1. What happens when you take successive differences?