Problem Solving Seminar

AMS Grad Student Chapter, University of Rochester

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Problem 1 (?*, contributed by Firdavs)**.** Determine all matrices $X \in \mathrm{Mat}_{3\times 3}(\mathbb{C})$ such that

$$X^2 = \begin{bmatrix} 2 & -4 & -3 \\ 1 & -4 & -4 \\ -1 & 5 & 5 \end{bmatrix}$$

Problem 2 (??, Putnam 2007)**.** Suppose that a finite group has exactly n elements of order p, where p is a prime. Prove that either $n = 0$ or p divides $n + 1$.

Problem 3 (++, USAMO?†)**.** Consider the integer lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$. We say that $(m, n) \in \mathbb{Z}^2$ is visible from the origin if the line segment joining $(0, 0)$ and (m, n) in $\mathbb{R}^2$ contains no points from the lattice.

- What fraction of the integer lattice (in the sense of natural density‡) is invisible from the origin?
- For arbitrary positive integers M and N, does there exists a rectangle in the lattice of dimensions $M \times N$ such that no lattice points inside the rectangle are visible from the origin? [1]

Problem 4 (???)**.** When viewed as locally compact (and Hausdorff) abelian groups under the usual group operations and the discrete topology, it is well known that $\mathbb{Z}/n\mathbb{Z}$ is dual to itself. In additive combinatorics, it is usual to normalize the Haar measure on $\mathbb{Z}/n\mathbb{Z}$ so that it is the counting measure on one side and the probability measure (i.e., the normalzied counting measure) on the other side, since this choice makes many normalization constants 1 (e.g. in Parseval-Plancherel).

Is there a deeper reason for this?

*+ indicates hardness; the more plusses there are, the harder I think the problem is. Conversely, − indicates easiness. ? indicates I don't know the solution, and the number of ?s indicates how hard I think the solution probably is.

†Part of this is from a mathematical olympiad I cannot recall, part of this is well-known mathematical folklore

‡Density of $A = \lim_{n\to\infty} \frac{|A \cap \{-n, \cdots, 0, \cdots, n\}^2|}{(2n+1)^2}$

Hints

1. Chinese Remainder Theorem