Problem Solving Seminar

AMS Grad Student Chapter, University of Rochester

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Problem 1 (++*, Lagrange's square theorem for a ring of polynomials)**.** Define $\mathcal{P}$ and $\mathcal{S}_n$ as follows:

$$\mathcal{P} = \{f \in \mathbb{R}[x] : \forall t \in \mathbb{R}, f(t) \geq 0\}$$

$$\mathcal{S}_n = \left\{ \sum_{j=1}^{n} f_j(x)^2 : f_j \in \mathbb{R}[x] \right\}$$

Further, define

$$\mathcal{S}_\infty = \bigcup_{n=0}^{\infty} \mathcal{S}_n$$

It is easy to see that $\mathcal{S}_n \subseteq \mathcal{P}$ and hence, $\mathcal{S}_\infty \subseteq \mathcal{P}$.

Is $\mathcal{P} = \mathcal{S}_\infty$? Furthermore, is there a positive integer k such that $\mathcal{S}_k = \mathcal{S}_\infty$? If yes, determine the smallest such k.

Problem 2 (??, from [1])**.** Let $\mathcal{P}$ be the convex hull in $\mathbb{R}^2$ of the points $\{p_1, \cdots, p_n\}$ with $p_j = (x_j, y_j)$. Define the stretched polygon $\mathcal{Q}$ associated with $\mathcal{P}$ to be the polygon obtained by stretching P along the x-axis with a factor $\alpha > 1$. Equivalently, Q is the convex hull of the points $\{q_1, \cdots, q_n\}$ with $q_j = (\alpha x_j, y_j)$.

Does there exist a rigid body motion that embeds $\mathcal{P}$ in $\mathcal{Q}$?

Problem 3 (+, Putnam†)**.** Let $f : \mathbb{N}^2 \to \mathbb{N}$ be an infinite grid of positive integers with the property that for every $x \in \mathbb{N}$,

$$|f^{-1}(k)| = \#\{(m,n) \in \mathbb{N}^2 : f(m,n) = k\} = 2019$$

Show that there exists positive integers m and n such that $f(m,n) > mn$. [1]

Problem 4 (+, Putnam 2008)**.** Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy? [2]

*+ indicates hardness; the more plusses there are, the harder I think the problem is. Conversely, − indicates easiness. ? indicates I don't know the solution, and the number of ?s indicates how hard I think the solution probably is.

†I don't have a citation for the problem year (I'm not even entirely convinced it was Putnam and not IMO); I've reproduced the problem with some superficial modifications from memory. If someone does know a reference, please let me know.

Hints

1. The property can be weakened to $\lim_{k\to\infty} \frac{|f^{-1}(k)|}{\log k} = 0$, and the stated result still holds.
2. The number 2008 can be replaced by 2020, but not by 2019.

References

[1] Dart (https://math.stackexchange.com/users/711241/dart). If i stretch a convex polygon, does the original fit into the streched version? Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/3380375 (version: 2019-10-04).