Problem Solving Seminar

AMS Grad Student Chapter, University of Rochester

25th November, 2019

Problem 1 (++*, contributed by Lucas)**.** Let $\mathcal{S}$ be a smallest class of functions $\mathbb{R}^2 \to \mathbb{R}$ (possibly with singularities) such that

- $(x, y) \mapsto x$ and $(x, y) \mapsto y$ are in $\mathcal{S}$.
- $f, g \in \mathcal{S} \implies f + g, f - g \in \mathcal{S}$
- $f \in \mathcal{S} \implies 1/f \in \mathcal{S}$

Does the map $(x, y) \mapsto 2019$ belong to $\mathcal{S}$?

Problem 2 (++, contributed by Firdavs)**.** Consider the vector space $\mathcal{V}$,

$$\mathcal{V} = \{(a_n)_{n=1}^{\infty} : a_n \in \mathbb{F}\}$$

the space of sequences over the field $\mathbb{F}$ with addition and scalar multiplication given by component-wise operations. Show that the Hamel basis of $\mathcal{V}$ over $\mathbb{F}$ is not countable. [1]

Problem 3 (????)**.** Is there a non-degenerate notion of a topological manifold over a finite field $\mathbb{F}_p$, or a topological manifold over the algebraic closure $\overline{\mathbb{F}_p}$?

Problem 4 (??, IMO 2009†)**.** Let $a_1, \cdots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at point 0 and making n jumps to the right with lengths $a_1, \cdots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

*+ indicates hardness; the more plusses there are, the harder I think the problem is. Conversely, − indicates easiness. ? indicates I don't know the solution, and the number of ?s indicates how hard I think the solution probably is.

†solved in a mini-polymath [Tao]

References

[Tao] Terry Tao. Imo 2009 q6 as a mini-polymath project. Terry Tao's blog. (version: 2019-11-25).

Hints

1. Probably obvious, but diagonalize (in both obvious senses of the word).