Problem Solving Seminar

AMS Grad Student Chapter, University of Rochester

28th October, 2019

Problem 1 (+*, USAMO 1991)**.** Show that, for any fixed integer $n \geq 1$, the sequence

$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \cdots \pmod{n}$$

is eventually constant.

Problem 2 (??, from [1])**.** Prove that every polynomial $f(x) \neq 0$ has a multiple $g(x) = f(x)h(x) \neq 0$ in which every exponent is prime. That is,

$$g(x) = \sum_p a_p x^p$$

where the sum is over primes.

Problem 3 (++, *Pairs at Maximum Distance* from Alex's copy of [2])**.** Let $X \subseteq \mathbb{R}^2$ be a finite set. Suppose X contains n points, and the maximum distance between any two of them is d. Prove that at most n pairs of points of X are at distance d. [1]

Problem 4 (???, translated from the Soviet Student Olympiads in Mathematics by Firdavs)**.** Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive real numbers such that

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \infty$$

Show that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx$$

is not Lebesgue integrable on $[0, 2\pi]$.

References

[1] László Babai. Reu 2012 - problems, puzzles problem sheet. Author's Webpage. http://people.cs.uchicago.edu/ laci/REU12/puzzles.pdf (version: 2019-10-28).

[2] Peter Winkler. *Mathematical puzzles: a connoisseur's collection.* AK Peters/CRC Press, 2003.

Hints

1. If $\{A, B\}$ and $\{C, D\}$ are pairs of points in X such that $AB = CD = d$, then what can you say about the line segments AB and CD?

*+ indicates hardness; the more plusses there are, the harder I think the problem is. Conversely, $-$ indicates easiness. ? indicates I don't know the solution, and the number of ?s indicates how hard I think the solution probably is.