Exercises for the Rochester AMS Grad Student Chapter Problem Solving Seminar

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(Note: The exercises below are independent of each other)

Let $n \in \mathbb{N}, n \geq 2$, and $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ with the surface measure σ defining integration on the unit sphere. A set $\Lambda \subset S^{n-1}$ is δ-separated for some $\delta > 0$ if $x, y \in \Lambda$ and $x \neq y$ implies $|x - y| \geq \delta$. The characteristic function is denoted by $1_Q(x) = 1$ if $x \in Q$ and 0 if $x \notin Q$.

The Stein-Thomas restriction conjecture [S] is

$$\|\widehat{f\sigma}\|_{L^p} \lesssim \|f\|_{L^2(\sigma)}, \quad p \geq \frac{2(n+1)}{n-1}. \tag{*}$$

An equivalent discrete version (see [BD]) is that for all $0 < \delta \leq 1$, for all $\Lambda \subset S^{n-1}$ sets which are $\delta^{1/2}$ separated, all $a_\xi \in \mathbb{C}$ and all balls B_R (of any center) where $R \sim \delta^{-1/2}$, the following holds

$$\left(\frac{1}{|B_R|} \int_{B_R} |\sum_{\xi \in \Lambda} e^{ix\cdot\xi} a_\xi|^p\right)^{1/p} \lesssim \delta^{\frac{n}{2p} - \frac{n-1}{4}} \|a_\xi\|_{l^2_\xi(\Lambda)}. \tag{**}$$

(Here $|B_R|$ denotes the volume of B_R)

1. Prove the exponent in (*) is necessary by applying (*) to

   $$f(x) = 1_{B(N,\delta)\cap S^{n-1}}(x).$$

   Here $N \in S^{n-1}$ is a point and $B(N, \delta)$ is a ball of radius δ centered at N. (Estimate f by 1_{Q_δ} where Q_δ is a $\delta \times \cdots \times \delta \times \delta^2$ rectangle; the Fourier transform of 1_{Q_δ} is "essentially supported" on a $\delta^{-1} \times \cdots \times \delta^{-1} \times \delta^{-2}$ rectangle).

2. For $p \geq \frac{2(n+1)}{n-1}$, conjugate q and some $C > 0$, show the equivalences:

   (a) $\|\widehat{f\sigma}\|_{L^p} \leq C\|f\|_{L^2(\sigma)}$.

   (b) $\|\widehat{f}\|_{L^2(\sigma)} \leq C\|f\|_{L^q}$.

   (c) $\|\widehat{\sigma} * f\|_{L^p} \leq C^2\|f\|_{L^q}$.

3. Show the equivalence between (*) and (**).

Remarks:

The inequality (**), although discrete is richer than the continuous counterpart (*). For example, looking at higher spatial scales $R \sim \delta^{-1}$, it is possible to obtain an improvement on the δ exponent,

$$\left(\frac{1}{|B_R|}\int_{B_R}|\sum_{\xi\in\Lambda} e^{ix\cdot\xi}a_\xi|^p\right)^{1/p} \lesssim \delta^{\frac{n+1}{2p}-\frac{n-1}{4}}\|a_\xi\|_{l^2_\xi(\Lambda)}. \qquad (\diamondsuit)$$

I found this fact, that a discretized version can be riched than its continuous counterpart, quite interesting and wanted to share it with you.

This improvement ($\diamondsuit$) is a direct application of the l^2 decoupling inequality recently proven in [BD]. For this and numerous other applications, see [BD].

References

[S] Stein, E. "Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals".

[BD] Bourgain, J., & Demeter, C. (2015). The proof of the l^2 Decoupling Conjecture. Annals of Mathematics, 182(1), second series, 351-389.