Homework 10

MA 35100 (Spring 2024, Section 130)

March 22nd, 2024

Instructions

- Due: Saturday, March 30th at 11 PM Eastern Time.
- Total Score: 25 points.
- The three lowest homework scores will be dropped from the final grade.
- One late submission is permitted (over the course of the semester) with no questions asked.
- Submissions can be hand-written or typed in LaTeX and must be submitted on Gradescope.
- You are allowed to discuss and collaborate on problems. However, each student must work on the final submission on their own. **In particular, copying someone else's final submission will be considered cheating and will be reported to the Office of the Dean of Students.**

Problem 0. [0 points] Copy paste the following text in the beginning of your submission:

> I have not made use of any unauthorized resources (including online resources) while working on this submission. Any collaboration with other students conforms with the policies of this course.

After that, list all students you collaborated with, clearly indicating which problems you worked with them on. If you did not collaborate with anyone, clearly state this instead.

Problem 1. [3 points] Suppose A and B are 3×3 matrices with the property that $\det A = 2$ and $\det B = -1$. Compute $\det(3A^3B^TA^{-1})$.

Problem 2. [6 points] Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

This is called the 3×3 Vandermonde determinant; you might see the $n \times n$ Vandermonde determinant in other courses; this is the determinant of the $n \times n$ matrix with (i, j)th entry equal to x_j^{i-1} for some numbers $x_1, \ldots, x_n$.

Find a similar formula for the 4×4 Vandermonde determinant, i.e., find

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}.$$

Problem 3. [8 points] Aliyah and Bahadur are playing a game on an empty 2024×2024 matrix A. Aliyah starts the game. In their turn, each player chooses an as-yet-unfilled position in the matrix and fills it with any real number they like. The game ends when every position in the matrix is filled out (that is, there are 2024^2 turns). Aliyah wins if, at the end, $\det A \neq 0$. Bahadur wins if $\det A = 0$. Who has a winning strategy in this game? Describe the strategy and explain why it works.

[**Hint:** The strategy I have in mind is extremely elegant and will work for any $n \times n$ matrix with n even, but will *not* work when n is odd.]

Problem 4. [8 points] Consider

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 3 & -1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Further, let $\vec{x}$ be the solution to $A\vec{x} = \vec{b}$. Using Cramer's rule, find x_2 and x_3. Further, find the $(2, 3)$ and $(1, 4)$ entries of A^{-1}.