Homework 13

MA 35100 (Spring 2024, Section 130)

April 15th, 2024

Instructions

- Due: Saturday, April 20th at 11 PM Eastern Time.
- Total Score: 25 points.
- The three lowest homework scores will be dropped from the final grade.
- One late submission is permitted (over the course of the semester) with no questions asked.
- Submissions can be hand-written or typed in LaTeX and must be submitted on Gradescope.
- You are allowed to discuss and collaborate on problems. However, each student must work on the final submission on their own. **In particular, copying someone else's final submission will be considered cheating and will be reported to the Office of the Dean of Students.**

Problem 0. [0 points] Copy paste the following text in the beginning of your submission:

> I have not made use of any unauthorized resources (including online resources) while working on this submission. Any collaboration with other students conforms with the policies of this course.

After that, list all students you collaborated with, clearly indicating which problems you worked with them on. If you did not collaborate with anyone, clearly state this instead.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 3 & 4 & 1 \\ 1 & 0 & 0 & 4 \\ 3 & 7 & 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 2 \\ 2 & 2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}.$$

Problem 1. [6 points] For each of the matrices $X \in \{A, B, C\}$ compute the LU decomposition.

Problem 2. [3 points] Solve the following system of equations in $\mathbb{C}$:

$$\begin{aligned}(1+i)z_1 - (1-i)z_2 + iz_3 &= 4i\\ z_1 + (2+3i)z_2 + 7z_3 &= 6+2i\\ iz_1 + z_2 - 7z_3 &= 0\end{aligned}$$

Problem 3. [3 points] Solve the following system of equations in $\mathbb{C}$:

$$\begin{aligned}(1+i)z_1 - (1-i)z_2 + z_3 &= 4i\\ z_1 - (2-i)z_2 + (7+i)z_3 &= 4i\\ iz_1 + z_2 - 7z_3 &= 0\end{aligned}$$

Problem 4. [13 points] At first glance, it may seem that real matrices having complex eigenvalues are an algebraic artifice introduced to make some calculations possible. In fact, the complex eigenvalues have a geometric interpretation, which we will explore now.

(a) Let $A \in M_n(\mathbb{R})$ be a $n \times n$ real matrix with a complex (strictly non-real) eigenpair $(\lambda, \vec{v})$. Show that the complex conjugate $(\overline{\lambda}, \overline{\vec{v}})$ must also be an eigenpair.

(b) Suppose now that $A \in M_2(\mathbb{R})$ has a strictly non-real eigenvalue $\lambda_1 = a + bi$ with eigenvector $\vec{v}_1 = \vec{u}_1 + i\vec{u}_2$ (here $a, b \in \mathbb{R}$ and $\vec{u}_j \in \mathbb{R}^2$). Conclude that A has another eigenpair, $(\lambda_2, \vec{v}_2)$ and write it in terms $a, b, \vec{u}_1, \vec{u}_2$. Thus, conclude that A is diagonalizable over $\mathbb{C}$.

(c) Argue that $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ is an ordered basis of $\mathbb{R}^2$. Let $B = [A]_{\mathcal{B}}$ be the matrix of A when written in the basis B. Show that

$$B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

(d) Let $r = |\lambda_1|$ and $\theta = \arg \lambda_1$. Show that

$$B = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

In other words, if A is a 2×2 real matrix with a strictly non-real eigenvalue λ then there is a change of basis after which it becomes clear that the action of A is a rotation by $\arg \lambda$ followed by a scaling by $|\lambda|$. Something similar is true for 3×3 matrices (in that case, it will be a rotation relative to some axis).