Homework 2

MA 35100 (Spring 2024, Section 130)

January 21st, 2024

Instructions

- Due: Tuesday, January 30th at 11 PM Eastern Time. Please note the unusual date.
- Total Score: 25 points.
- The three lowest homework scores will be dropped from the final grade.
- One late submission is permitted (over the course of the semester) with no questions asked.
- Submissions can be hand-written or typed in LaTeX and must be submitted on Gradescope.
- You are allowed to discuss and collaborate on problems. However, each student must work on the final submission on their own. **In particular, copying someone else's final submission will be considered cheating and will be reported to the Office of the Dean of Students.**

Problem 0. [0 points] Copy paste the following text in the beginning of your submission:

> I have not made use of any unauthorized resources (including online resources) while working on this submission. Any collaboration with other students conforms with the policies of this course.

After that, list all students you collaborated with, clearly indicating which problems you worked with them on. If you did not collaborate with anyone, clearly state this instead.

Problem 1. [12 points] For the following sets, determine if they are closed under addition. If they aren't, give concrete examples showing the failure of closure and if they are give a brief justification. Then, do the same for closure under scalar multiplcation.

1. The set of vectors (x, y) in $\mathbb{R}^2$ satisfying the equations $xy = 0$.
2. The set of all polynomials $p(x)$ with the property that $p(0)$ is an integer.
3. The set of vectors (x, y) in $\mathbb{R}^2$ satisfying $y = x^2$.

4. The set of functions $f : [-1,1] \to \mathbb{R}$ with the property that $f(-1) = f(1)$.

Problem 2. [4 points] For the vector space $V = \mathbb{R}^4$, determine if the vector $v = (3, 0, 2, 2)$ is dependent on the set $S = \{v_1, v_2, v_3\}$. If it is, find an explicit linear dependence.

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ -3 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -3 \\ 2 \\ 3 \\ -2 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} -2 \\ -2 \\ 3 \\ -2 \end{bmatrix}.$$

Problem 3. [4 points] For the vector space $V = \mathbb{R}^3$, determine if the vector $v = (1, 2, 3)$ is dependent on the set $S = \{v_1, v_2, v_3\}$. If it is, find an explicit linear dependence.

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}.$$

Problem 4. [4 points] For the vector space $V = P_5(R)$, determine if the vector polynomial $p(t) = 1 + t^5$ is dependent on the set $S = \{p_1, p_2, p_3, p_4\}$. If it is, find an explicit linear dependence.

$$p_1(t) = 1 + t + t^5, \qquad p_2(t) = 1 - t + 3t^2 + t^5,$$
$$p_3(t) = 3t^2 + t^4, \qquad p_4(t) = -1 + t^4 - t^5.$$

Problem 5. [1 points] One intuitive way to define "dimension" is "degrees of freedom" – i.e., the number of real numbers that need to be specified to determine an element of a vector space. Using this intuitive definition, what do you think is a reasonable value for the dimensions of the following spaces? You do not need to justify your answer.

$$V_1 = \mathbb{R}^4, \qquad V_2 = P_2(\mathbb{R}), \qquad V_3 = \mathcal{C}[0,1], \qquad V_4 = \{A \in M_{2\times 2}(\mathbb{R}) : A^T = A\}$$

Note: Here A^T refers to the transpose matrix. If A is an $m \times n$ matrix, then $B = A^T$ is $n \times m$ matrix such that $b_{ij} = a_{ji}$ for all $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$.