Homework 9

MA 35100 (Spring 2024, Section 130)

March 6th, 2024 (Updated: March 19th)

Instructions

- Due: Saturday, March 23rd at 11 PM Eastern Time.
- Total Score: 25 points.
- The three lowest homework scores will be dropped from the final grade.
- One late submission is permitted (over the course of the semester) with no questions asked.
- Submissions can be hand-written or typed in LaTeX and must be submitted on Gradescope.
- You are allowed to discuss and collaborate on problems. However, each student must work on the final submission on their own. **In particular, copying someone else's final submission will be considered cheating and will be reported to the Office of the Dean of Students.**

Problem 0. [0 points] Copy paste the following text in the beginning of your submission:

> I have not made use of any unauthorized resources (including online resources) while working on this submission. Any collaboration with other students conforms with the policies of this course.

After that, list all students you collaborated with, clearly indicating which problems you worked with them on. If you did not collaborate with anyone, clearly state this instead.

Problem 1. [6 points] Consider set $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ given by

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \vec{b}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

Check that $\mathcal{B}$ is an (ordered) basis for $\mathbb{R}_3$. Compute the point matrix $P_{\mathcal{B}}$ and the coordinate matrix $C_{\mathcal{B}}$.

Further, let $\vec{x}, \vec{y} \in \mathbb{R}^3$ be such that

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad [\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Compute $[\vec{x}]_{\mathcal{B}}$ and $\vec{y}$.

Problem 2. [5 points] Recall that the map $D : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ given by $D(p) = p'$ is a linear transformation. Compute the matrix of this transformation in the standard ordered basis $\mathcal{B} = \{1, t, t^2, \cdots, t^n\}$ of $P_n(\mathbb{R})$.

Problem 3. [7 points] We will show that[1] if $n \neq m$, then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not isomorphic to each other. In fact, if one only cares about finite-dimensional vector spaces up to isomorphism, then for each dimension n there is a unique vector space, namely $\mathbb{R}^n$.

(a) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an injective linear transformation. Show that, in this case, $n \leqslant m$.

(b) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a surjective linear transformation. Show that, in this case, $n \geqslant m$.

(c) Conclude that if $T : \mathbb{R}^n \to \mathbb{R}^m$ is an isomorphism, then $n = m$.

(d) Let V be a finite-dimensional vector space with dimension n. Show that V is isomorphic to $\mathbb{R}^n$.

(e) Conclude that if V and W are finite-dimensional vector spaces of the same dimension, then they are isomorphic to each other and also isomorphic to a unique Euclidean space with the same dimension.

Hints: For (a) and (b), try computing the rank of the matrix of T. For (d), try to think of some natural map you know from V to $\mathbb{R}^n$ or vice-versa.

Problem 4. [7 points] Consider

$$V = \{f \in \mathcal{D}^\infty(\mathbb{R}) : f''(t) + f(t) = 0 \text{ for all } t \in \mathbb{R}\}.$$

One can check that V is a vector space[2]. We have a map $T : V \to \mathbb{R}^2$ given by

$$T(f) = (f(0), f'(0)).$$

Check that T is a linear transformation. In fact, we will show[3] that T is an isomorphism.

(1) Show that if $f_1(t) = \cos t$ and $f_2(t) = \sin t$, then $f_1, f_2 \in V$.

(2) Show that f_1 and f_2 are linearly independent. **Hint:** let $f = c_1 f_1 + c_2 f_2$ be a linear combination such that $f \equiv 0$ and then compute $T(f)$.

[1] Note that this also follows from two things we proved in class: $\dim(\mathbb{R}^n) = n$ and the fact that dimension is an isomorphism-invariant – i.e., if $T : V \to W$ is an isomorphism, then $\dim V = \dim W$.

[2] You do not need to show this, though you should convince yourself it can be done.

[3] Up to an appeal to the uniqueness theorem for linear ODEs.

(3) In fact, it is the case that $\{f_1, f_2\}$ is spanning. This is implied by a deep fact from the theory of ordinary differential equations, namely the uniqueness of linear ordinary differential equations. We will just assume this.

(4) Conclude that $\dim V = 2$, and hence (why?) that T is an isomorphism by showing that T is surjective.

(5) Finally, compute the matrix of T in the basis $\mathcal{B} = \{f_1, f_2\}$ of V and the standard basis of $\mathbb{R}^2$.