MA 35100 (SPRING 2024, SEC: 130)

ANURAG SAHAY
email: anuragsahay@purdue.edu

Zoom ID: 995-911-5715

SECTION WEB PAGE https://www.math.purdue.edu/~sahay5/spring2024/ma35100/index.html

NOTE: ALL IMAGES ARE FROM THE TEXTBOOK (R.C. PENNEY, 4th ed.)

ANNOUNCEMENTS / NOTES

1. CLASSES RESUME IN-PERSON ON WEDNESDAY.
2. HW 5 IS DUE ON SATURDAY, 17th FEBRUARY
3. OFFICE HOURS ON ThF 2:30 PM - 4:00 PM THIS WEEK.
4. RECORDING OF CLASS WILL BE UPLOADED TO KALTURA.
5. FINAL EXAM SCHEDULE : 30th APRIL (7:00 - 9:00 PM), PHYS 203.
   (TUESDAY)
6. TRY TO KEEP VIDEOS ON.

RECALL :

$$\text{Null}(A) = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \}$$

$$\text{NULLITY} = \dim(\text{Null}(A))$$

Q. HOW TO FIND A BASIS FOR Null(A)?

$\Updownarrow$

SOLVING $A\vec{x} = \vec{0}$ $\rightsquigarrow$ $\vec{x} = c_1 \vec{b}_1 + \cdots + c_k \vec{b}_k$

$k = \boxed{n - r} \rightarrow$ # OF FREE VARS.

n: # OF VARS.

r: RANK = # OF CONSTRAINTS.

e.g. :

$$A = \begin{bmatrix} 2 & 4 & -3 & 5 \\ 1 & 1 & 1 & 1 \\ 4 & 6 & -1 & 7 \end{bmatrix} \xrightarrow{\text{EROs}} \text{(R.R)E.F.}$$

$$\text{Null}(A) \equiv [A \mid \vec{0}] \qquad (A\vec{x} = \vec{0})$$

$$A \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 2 & 4 & -3 & 5 \\ 4 & 6 & -1 & 7 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \boxed{2} & -5 & 3 \\ 0 & 2 & -5 & 3 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$n = 4,\ r = 2$

$$x_1 + x_2 + \boxed{x_3} + \boxed{x_4} = 0$$

$$2x_2 \boxed{-5x_3} + \boxed{3x_4} = 0$$

$$x_1 + x_2 + \boxed{x_3} + \boxed{x_4} = 0$$

$$2x_2 - \boxed{5x_3} + \boxed{3x_4} = 0$$

$$x_3 = 2s, \quad x_4 = 2t$$

$$x_2 = \frac{5x_3 - 3x_4}{2} = 5s - 3t$$

$$x_1 = -x_2 - x_3 - x_4 = (-5s + 3t) - 2s - 2t = -7s + t$$

$A\vec{x} = \vec{0} \Rightarrow$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7s + t \\ 5s - 3t \\ 2s \\ 2t \end{bmatrix} = \begin{bmatrix} -7s \\ 5s \\ 2s \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ -3t \\ 0 \\ 2t \end{bmatrix} = s \begin{bmatrix} -7 \\ 5 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$$

BASIS FOR Null(A) $= \left\{ \begin{bmatrix} -7 \\ 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} \right\}.$

NULLITY $= 2$

IN FACT, NULLITY $=$ # OF FREE VARIABLE

RECALL:

IF $\vec{x}_0$ IS A SOLN. TO $A\vec{x} = \vec{b}$
THEN THE GENERAL SOLN. IS
GIVEN BY

$$\vec{x}_0 + \mathrm{Null}(A) \cong \vec{x}_0 + \vec{z}, \quad \vec{z} \in \mathrm{Null}(A)$$

$\therefore$ NULLITY MEASURES THE "SIZE" OF THE SOLN SET OF $A\vec{x} = \vec{b}$!

Thm (RANK-NULLITY)

FOR AN $m \times n$ MATRIX, A

$$\underbrace{\text{RANK}}_{\dim(\text{Col}(A)) = \dim(\text{Row}(A))} + \underbrace{\text{NULLITY}}_{\dim(\text{Null}(A))} = n$$

e.g.: FOR PVS. EXAMPLE

$n = 4$, $r = 2$, $n = 2$ $(2 + 2 = 4)$.

Pf

RANK = # OF PIVOT COLUMNS

NULLITY = # OF NON-PIVOT COLUMNS.

(= # OF FREE VARIABLES $= n - \lambda$)

$\Rightarrow$ n = RANK + NULLITY

(# OF COLS.) (# OF PIVOT COLS.) (# OF NON-PIVOT COLS.)

e.g.

$$B = \begin{bmatrix} 2 & 1 & 8 \\ 1 & 0 & 0 \\ 0 & 0 & 7 \\ 0 & 1 & 0 \end{bmatrix}$$

$n = 3$ (# OF COLS / VARS)

$r = 3$ (# OF PIVOT / CONSTRAINTS)

$n - r = 0$ (# OF NON-PIVOT / DEGREES OF FREEDOM)

$$\Big\downarrow \; ① \; R_1 \leftrightarrow R_2, \; ② \; R_2 \leftrightarrow R_4$$

① $R_4 \to R_4 - 2R_1$

② $R_4 \to R_4 - R_1$

③ $R_3 \to \frac{1}{7} R_3$

④ $R_4 \to R_4 - 8R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \;\cancel{7} \\ 0\;\cancel{2} & 0\;\cancel{1} & \cancel{8}\;0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Updownarrow$$

$$x_1 = x_2 = x_3 = 0$$

$A\vec{x} = \vec{b}$

Theorem 2.18 *Let A be an $m \times n$ matrix. There is at most one solution to* ~~$AX = B$~~ *for all* ~~B~~ $\vec{b}$ $\in \mathbb{R}^m$ *if and only if rank* $A = n$.

$\Longleftarrow$

"FULL RANK"

RANK-NULLITY $\Rightarrow$ NULLITY $= n -$ RANK $= 0$

$\Rightarrow \text{Null}(A) = \{\vec{0}\}$.

$\therefore$ FOR EVERY $\vec{b} \in \mathbb{R}^m$, EITHER:

① NO SOLNS FOR $A\vec{x} = \vec{b}$. ($[A \mid \vec{b}]$ IS INCONSISTENT)

② UNIQUE (EXACTLY ONE) SOLN TO $A\vec{x} = \vec{b}$ ($[A \mid \vec{b}]$ IS CONSISTENT).

$A\vec{x} = \vec{b}$

Theorem 2.18 *Let A be an $m \times n$ matrix. There is at most one solution to* ~~$AX = B$~~ *for all* ~~B~~ $\vec{b} \in \mathbb{R}^m$ *if and only if rank* $A = n$.

$\Rightarrow$

CONSIDER $\vec{b} = 0$.

$A\vec{x} = \vec{0}$

$\boxed{\vec{x} = 0} \to$ UNIQUE SOLN!

$\text{Null}(A) = \{\vec{0}\}$ $\Rightarrow$ NULLITY $= 0$

$\Rightarrow$ RANK $= n -$ NULLITY $= n$

NONSINGULAR MATRICES

Definition 2.8 *An $m \times n$ matrix A is said to be* **nonsingular** *if the solution to* $\cancel{AX = B}$ *both exists and is unique for all* $\cancel{B \in \mathbb{R}^n}$.

$A\vec{x} = \vec{b}$ $\qquad$ $\vec{b} \in \mathbb{R}^m$

PRUS THM IMPLIES THAT NONSINGULAR MATRICES ARE FULL RANK.

IDENTITY MATRIX

$$A = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{n \times n}$$

CHECK NONSINGULAR!

Theorem 2.19 *The matrix A is nonsingular if and only if A is an $n \times n$ matrix with rank n.*

Pf: $(\Leftarrow)$ $A \to n \times n$, RANK $= n$

$$\mathrm{Col}(A) \subseteq \mathbb{R}^n$$

$$\dim(\mathrm{Col}(A)) = n = \dim(\mathbb{R}^n)$$

$$\Rightarrow \quad \mathrm{Col}(A) = \mathbb{R}^n$$

$$\{\vec{b} \in \mathbb{R}^n : A\vec{x} = \vec{b} \text{ HAS A SOLN}\}$$

$\therefore$ EVERY $A\vec{x} = \vec{b}$ IS $\boxed{\text{UNIQUELY}}$ SOLVABLE!

($\Rightarrow$) I. IF RANK $\neq n \Rightarrow A\vec{x} = \vec{b}$ HAS ∞LY MANY SOL'NS FOR AT LEAST ONE $\vec{b} \in \mathbb{R}^m$

$\Rightarrow$ A IS SINGULAR.

II. IF $m \neq n$, RANK $= n$
$\leq m$

$\Rightarrow n < m$

$\vec{b} \notin \underset{\dim n}{\mathrm{Col}(A)} \subseteq \underset{\dim m}{\mathbb{R}^m} \ni \vec{b}$