MA 35100 (SPRING 2024, SEC: 130)

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SECTION WEB PAGE https://www.math.purdue.edu/~sahay5/spring2024/ma35100/index.html

NOTE: ALL IMAGES ARE FROM THE TEXTBOOK (R.C. PENNEY, 4th ed.)

ANNOUNCEMENTS/NOTES

1. FRIDAY CLASS ALSO ON ZOOM.

2. NO OFFICE HOURS OR H.W. THIS WEEK

¡ PLEASE WORK ON H.W. 2 NOW ANYWAY ! (DUE: TUESDAY, JAN 30th)

3. RECORDING OF CLASS WILL BE UPLOADED TO KALTURA.

4. TRY TO KEEP VIDEOS ON.

RECALL:

DEFN: A SET S IS LINEARLY INDEPENDENT (L.I.) IF NONE OF THE ELEMENTS IS A LINEAR COMBINATION OF THE OTHERS.

ISSUE: HARD TO CHECK!

e.g. $S = \{v_1, \cdots, v_{100}\}$?

1. $\vec{v}_1 \overset{?}{=} x_2 \vec{v}_2 + \cdots + x_{100} \vec{v}_{100}$
2. $\vec{v}_2 \overset{?}{=} x_1 \vec{v}_1 + x_3 \vec{v}_3 + \cdots + x_{100} \vec{v}_{100}$

$\vdots$

100. ______________

TEST

Theorem 2.1 (Test for Independence). *Let $S = \{A_1, A_2, \ldots, A_n\}$ be a set of n elements of a vector space $\mathcal{V}$. Consider the equation*

DEPENDENCE EQUATION ←

$$\boxed{x_1A_1 + \cdots + x_nA_n = \mathbf{0}} \tag{2.1}$$

If the only solution to this equation is $x_1 = x_2 = \cdots = x_n = 0$, then S is linearly independent.

On the other hand, if there is a solution to equation (2.1) with $x_k \neq 0$ for some k, then S is linearly dependent and A_k is a linear combination of the set of A_j with $j \neq k$.

$$\underbrace{x_1}_{}A_1 + \underbrace{x_2}_{}A_2 + \cdots + \underbrace{x_n}_{}A_n = 0$$

TRIVIAL SOLN. ← $\boxed{x_j = 0}$

$$0 \cdot A_1 + \cdots + 0 \cdot A_n = 0$$

S IS L.I. $\Longleftrightarrow$ $\{ x_1 A_1 + \cdots + x_n A = 0$ HAS NO NON-TRIVIAL SOLNs.

1) SUPPOSE S IS L.D.

$$A_1 = c_2 A_2 + c_3 A_3 + \cdots + c_n A_n$$

$$\boxed{-A_1 + c_2 A_2 + \cdots + c_n A_n = 0}$$

$\boxed{x_1 = -1}$, $x_2 = c_2$, $x_3 = c_3$, $\cdots$, $x_n = c_n$

$\neq 0$

2) SUPPOSE $\boxed{x_1} A_1 + \boxed{x_2 A_2 + \cdots + x_n A_n} = 0$

HAS A NON-TRIVIAL SOLN.

$x_1 \neq 0$

$$A_1 = \left(\frac{-x_2}{x_1}\right) A_2 + \left(\frac{-x_3}{x_1}\right) A_3 + \cdots \left(\frac{-x_n}{x_1}\right) A_n$$

A_1 IS L.D. ON $\{A_2, \ldots, A_n\}$

$\therefore$ S IS L.D.

e.g. $V = \mathbb{R}^3$

$$\boxed{S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}}$$

$v_1 = (1, 1, 2)$

$v_2 = (1, 2, 1)$

$v_3 = (0, 1, 1)$

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = 0$$

$$\downarrow$$

$$x_1\begin{bmatrix}1\\1\\2\end{bmatrix} + x_2\begin{bmatrix}1\\2\\1\end{bmatrix} + x_3\begin{bmatrix}0\\1\\1\end{bmatrix} = 0$$

$$\boxed{\text{ANS: L.I!}}$$

$$x_1\begin{bmatrix}1\\1\\2\end{bmatrix} + x_2\begin{bmatrix}1\\2\\1\end{bmatrix} + x_3\begin{bmatrix}0\\1\\1\end{bmatrix} = 0$$

$$= \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 + x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left.\begin{array}{l} x_1 + x_2 = 0 \\ x_1 + 2x_2 + x_3 = 6 \\ 2x_1 + x_2 + x_3 = 0 \end{array}\right\} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & \underline{-1} & 1 & 0 \end{array}\right]$$

$$\Big\downarrow R_3 \to R_3 + R_2$$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]}_{x_1 = x_2 = x_3 = 0} \xleftarrow[R_1 \to R_1 - R_2]{} \xleftarrow{R_2 \to R_2 - R_3} \xleftarrow{R_3 \to \frac{1}{2} R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

■ EXAMPLE 2.3

Test the set formed by the following matrices for linear dependence. If linear dependence is found, use your results to express one as a linear combination of the others.

$$\overset{A_1}{\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}}, \quad \overset{A_2}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}, \quad \overset{A_3}{\begin{bmatrix} -1 & -2 \\ -3 & -5 \end{bmatrix}}, \quad \overset{A_4}{\begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}}$$

$V = M_{2\times 2}(\mathbb{R})$

$$x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4 = 0$$

$$\Rightarrow x_1 \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + x_3 \begin{bmatrix} -1 & -2 \\ -3 & -5 \end{bmatrix} + x_4 \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$x_1\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix} + x_2\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} + x_3\begin{bmatrix}-1 & -2\\ -3 & -5\end{bmatrix} + x_4\begin{bmatrix}-1 & -2\\ 0 & -2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\ni \begin{bmatrix}x_1+x_2-x_3-x_4 & 2x_1+2x_2-2x_3-2x_4\\ x_1+2x_2-3x_3 & 3x_1+4x_2-5x_3-2x_4\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$x_1 + x_2 - x_3 - x_4 = 0$$

$$2x_1 + 2x_2 - 2x_3 - 2x_4 = 0$$

$$x_1 + 2x_2 - 3x_3 = 0$$

$$3x_1 + 4x_2 - 5x_3 - 2x_4 = 0$$

$$\left[\begin{array}{cccc|c}1 & 1 & -1 & -1 & 0\\ 2 & 2 & -2 & -2 & 0\\ 1 & 2 & -3 & 0 & 0\\ 3 & 4 & -5 & -2 & 0\end{array}\right]$$

PIVOT?

$$\left[\begin{array}{cccc|c}1&1&-1&-1&0\\2&2&-2&-2&0\\1&2&-3&0&0\\3&4&-5&-2&0\end{array}\right]\xrightarrow{R_2 - R_2 - 2R_1}\left[\begin{array}{cccc|c}1&1&-1&-1&0\\ \hline 0&0&0&0&0\\ \hline 1&2&-3&0&0\\3&4&-5&-2&0\end{array}\right]$$

r

$r^\#$

$n = 4$

$r = r^\#$

$r \leq 3$

$r^\# = r \leq 3 < 4 = n$ $\Rightarrow$ ∞LY MANY SOLNS.

IF $V = \mathbb{R}^m$, $S = \{v_1, \dots, v_n\}$

THEN THE AUGMENTED MATRIX OF THE DEPENDENCY RELATION HAS A THE SPECIAL FORM

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n = \vec{0}$$

$$x_1 \begin{bmatrix} \vec{v}_1 \\ \\ \\ \end{bmatrix} + x_2 \begin{bmatrix} \vec{v}_2 \\ \\ \\ \end{bmatrix} + \cdots + x_n \begin{bmatrix} \vec{v}_n \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \\ \\ \end{bmatrix}$$

AUGMENTED MATRIX $\longrightarrow$

$$\left[\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n & \vec{0} \\ \\ \end{array}\right] = \left[\begin{array}{cccc|c} \overset{(v_1)_1}{\square} & \overset{(v_2)_1}{\square} & \cdot & \cdot & 0 \\ \square & & & & \\ \\ \end{array}\right]$$

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

$$x_1\begin{bmatrix}1\\1\\2\end{bmatrix} + x_2\begin{bmatrix}1\\2\\1\end{bmatrix} + x_3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&0&0\\1&2&1&0\\2&1&1&0\end{array}\right]$$

$V = \mathbb{R}^m \qquad S = \{\vec{v}_1, \dots, \vec{v}_n\}$

DEFN.: PIVOT COLUMN / PIVOT VECTOR.

$$\text{DEPENDENCY EQN.} \xrightarrow[\text{MATRIX}]{\text{AUG.}} \left[\begin{array}{ccc|c} \vec{v}_1 & \cdots & \vec{v}_n & \vec{0} \end{array}\right]$$

$$\Big\downarrow \text{EROs}$$

$$\left[\begin{array}{cccc|} \boxed{1} & * & * & * \\ 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

COLUMNS/VECTORS WHICH END UP HAVING THE PIVOT ARE CALLED PIVOT COLUMNS/VECTORS

$(V = \mathbb{R}^m)$

Thm: THE SUBSET OF PIVOT VECTORS IN $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ IS L. I.

FURTHER, EVERY NON-PIVOT VECTOR IS DEPENDENT ON THE SET OF PIVOT VECTORS.

$n = 4$.

$$\left[\ \underset{\text{PIVOT}}{\boxed{\vec{v}_1}}\ \ \overset{\text{non-pivot}}{\boxed{\vec{v}_2}}\ \ \underset{\text{PIVOT}}{\boxed{\vec{v}_3}}\ \ \vec{v}_4\ \middle|\ \vec{0}\ \right]$$

$$x_1\vec{v}_1 + x_3\vec{v}_3 = 0$$

$$\Updownarrow$$

$$x_1 = x_3 = 0$$

$$\left[\begin{array}{cccc|c} \underline{\vec{v}_1} & \vec{v}_2 & \underline{\vec{v}_3} & \vec{v}_4 & \vec{0} \end{array}\right] \xrightarrow{\text{ERO's}} \left[\begin{array}{cccc|c} 1 & * & * & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} \vec{v}_1 & \vec{v}_3 & \vec{0} \end{array}\right] \xrightarrow{\text{ERO's}} \begin{array}{cc} x_1 & x_3 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 \vec{v}_1 + x_3 \vec{v}_3 = \vec{0}$$

$$x_1 = 0$$

$$x_3 = 0$$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \xrightarrow{\text{EROs}} \underbrace{\left[\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{RANK} = 2}$$

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$$

$$\boxed{r^{\#} = 2 < n} \rightarrow \infty\text{LY MANY SOLNs.}$$

$$x_2 \neq 0$$

$$\Rightarrow \quad v_2 = -\frac{x_1 v_1}{x_2} - \frac{x_3 v_3}{x_2}$$