MA 35100 (SPRING 2024, SEC: 130)

ANURAG SAHAY
email: anuragsahay@purdue.edu

Zoom ID: 995 - 911 - 5715

SECTION WEB PAGE https://www.math.purdue.edu/~sahay5/spring2024/ma35100/index.html

NOTE: ALL IMAGES ARE FROM THE TEXTBOOK (R.C. PENNEY, 4th ed.)

ANNOUNCEMENTS / NOTES

1. CLASSES RESUME IN-PERSON NEXT WEEK.

2. NO OFFICE HOURS OR H.W. THIS WEEK

   ¡ PLEASE WORK ON H.W. 2 NOW ANYWAY ! (DUE: TUESDAY, JAN 30th)

3. RECORDING OF CLASS WILL BE UPLOADED TO KALTURA.

4. TRY TO KEEP VIDEOS ON.

RECALL:

DEFN: IF $S = \{\vec{v}_1, \dots, \vec{v}_k\}$, $(v_j \in V)$
THEN

$$\text{Span } S = \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k : c_j \in \mathbb{R}\}$$

IS THE SET OF LINEAR COMBINATIONS USING S.

e.g. $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

e.g. $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

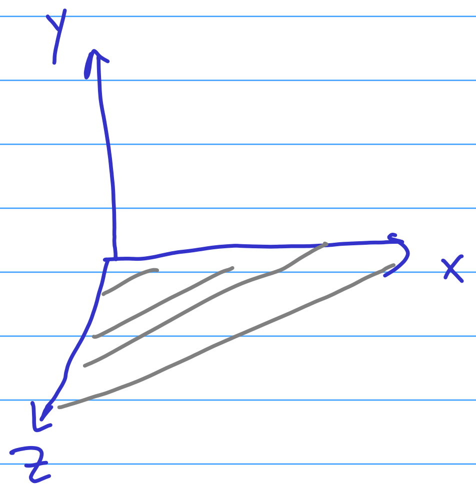
$$\underbrace{c_1}_{x} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \underbrace{c_2}_{y} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \underbrace{c_3}_{z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\mathbb{R}^3$

e.g. $V = \mathbb{R}^3$ $\qquad S = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

$$\text{Span } S \longrightarrow \underset{x}{\boxed{c_1}}\begin{bmatrix}1\\0\\0\end{bmatrix} + \underset{z}{\boxed{c_2}}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}c_1\\ \boxed{0}\\ c_2\end{bmatrix}$$

$\mathbb{R}^3$



$$\begin{bmatrix}x\\0\\z\end{bmatrix}$$

$$\neq 0 \qquad \begin{bmatrix}x\\ \boxed{y}\\ z\end{bmatrix} \overset{?}{=} c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\0\\1\end{bmatrix}$$

DEFN : $A \in M_{m \times n}(\mathbb{R})$

TRANSPOSE $\longrightarrow$ $A^T \in M_{n \times m}(\mathbb{R})$

$$a_{ij}^T = a_{ji} \quad \left(\begin{matrix} 1 \leq i \leq n \\ 1 \leq j \leq m \end{matrix} \right)$$

$$\underset{(2\times 2)}{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \underset{(2\times 2)}{A^T} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\underset{(2\times 3)}{A} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \qquad \underset{(3\times 2)}{A^T} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}$$

$A \to$ SQUARE MATRIX $M_{n \times n}(\mathbb{R})$

$A^T \in M_{n \times n}(\mathbb{R})$

IS $A = A^T$?

<u>DEFN</u>: IF $A = A^T$, THEN A IS CALLED A SYMMETRIC MATRIX.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \ne \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

e.g. $V = M_{2\times 2}(\mathbb{R})$ $\quad S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

$y \neq z$

$\neq$

$$\operatorname{Span} S \cong \quad c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & \boxed{c_2} \\ \boxed{c_2} & c_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \boxed{2} \\ \boxed{3} & 4 \end{bmatrix}$$

$$\text{Span } S = \left\{ \boxed{\begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix}} : c_1, c_2, c_3 \in \mathbb{R} \right\}$$

$\rightarrow$ SYMMETRIC

Q: IS THERE A 2x2 SYMMETRIC MATRIX "B"

s.t.

$$B \notin \text{Span } S?$$

NO! $\rightarrow$ $B = \begin{bmatrix} \boxed{b_{11}} & \boxed{b_{12}} \\ \boxed{b_{21}} & \boxed{b_{22}} \end{bmatrix} = B^T = \begin{bmatrix} \boxed{b_{11}} & \boxed{b_{21}} \\ \boxed{b_{12}} & \boxed{b_{22}} \end{bmatrix}$

$b_{12} = b_{21}$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix}$$

$c_1 = b_{11}$ $\quad$ $c_2 = b_{12}$ $\quad$ $c_3 = b_{22}$

Span S = SET OF 2×2 SYMMETRIC MATRICES.

$$\boxed{\text{e.g.} \quad V = P_2(\mathbb{R}) \qquad S = \{t,\ t^2\}}$$

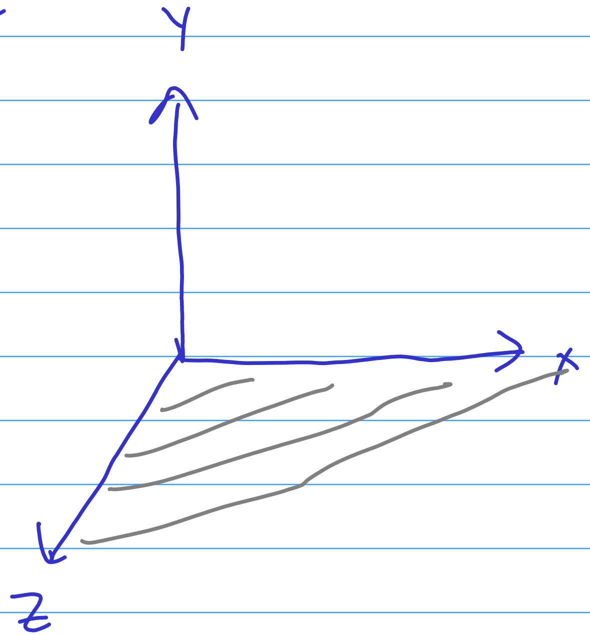
$\to$ H.W. ?

e.g. $V = \mathbb{R}^3$ $\quad S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$V = \mathbb{R}^3$

$$\operatorname{Span} S \cong c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 + c_3 \\ 0 \\ c_2 + c_3 \end{bmatrix}$$



$$\begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} c_1 + c_3 \\ 0 \\ c_2 + c_3 \end{bmatrix} = \begin{bmatrix} x - z + z \\ 0 \\ 0 + z \end{bmatrix}$$

$(x-z)$

$c_1 = x - z$

$c_2 = 0$

$c_3 = z$

OR

$c_1 = x$

$c_2 = z$

$c_3 = 0$

$\neq 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} c_1 + c_3 \\ 0 \\ c_2 + c_3 \end{bmatrix}$$

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = x-z \text{ PLANE}$$

$$= \text{Span}\overbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}}^{\text{L.D.}}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = v_1 + v_2$$

$$\text{Span}\{v_1, v_2, v_3\} \cong \boxed{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 + \vec{v}_2)$$

$$= \underbrace{(c_1 + c_3)}_{c_1'}\vec{v}_1 + \underbrace{(c_2 + c_3)}_{c_2'}\vec{v}_2$$

$$= \boxed{c_1'\vec{v}_1 + c_2'\vec{v}_2}$$

$$\cong \text{Span}\{v_1, v_2\}$$

Thm : LET $\vec{v}_1, \ldots, \vec{v}_k \in V$

s.t. $v_k \in \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\} \Leftrightarrow \vec{v}_k$ IS L.D. ON $\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$

THEN,

$$\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$$

(e.g. $V = \mathbb{R}^3$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = v_2 + v_3$

$\in \text{Span}\{v_1, v_2\}$.

~~Pf~~ . $c_1 v_1 + \cdots + c_{k-1} v_{k-1} + c_k \boxed{v_k}$

WRITE AS A L.C. OF $\{v_1, \dots, v_{k-1}\}$

$\rightsquigarrow$ A LINEAR COMBINATION OF $\{v_1, \dots, v_{k-1}\}$.

Q WHEN IS $\vec{b} \in \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$?

$\longrightarrow$ $\text{Span}\{a_1, \ldots, a_n\} \equiv$ SOLUTION SET

$\vec{b}$

EQUIVALENT TO SOLVING $\longrightarrow$ $\vec{b} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n$

WITH VARIABLES $x_1, \ldots, x_n \in \mathbb{R}$

SKIP

$V = \mathbb{R}^3$

e.g.

$$(13, -16, 1) \in \operatorname{Span}\left\{ \begin{array}{l} (2, -3, 1) \\ (-1, 1, 1) \\ (-3, 4, 0) \end{array} \right\}$$

~~SKIP~~

$V = P_2(\mathbb{R})$

e.g. IS $q \in \text{Span}\{p_1, p_2, p_3\}$

$q(x) = -3 + 14x^2$

$p_1(x) = 2 - x + 3x^2$

$p_2(x) = 1 + x + x^2$

$p_3(x) = -5 + 4x + x^2$