MA 351 LECTURE NOTES: FORMULAE FOR FIBONACCI NUMBERS USING THE GOLDEN RATIO

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1. Introduction

The so-called *golden ratio* φ is defined as follows: if you break a line segment into two parts so that the ratio of the whole to the bigger part is proportional to the ratio of the bigger part to the smaller part, then the segment is divided into the golden ratio. Thus, if a is the bigger part and b is the smaller part,

$$\varphi = \frac{a+b}{a} = \frac{a}{b}.$$

In several artistic fields (most notably, architecture) people incorporated the golden ratio into their works, especially during the Renaissance. From the definition, we see that

$$\varphi = 1 + \frac{b}{a} = 1 + \frac{1}{\varphi},$$

and with some rearrangement that $\lambda = \varphi$ must be a solution to the polynomial

$$p(\lambda) = \lambda^2 - \lambda - 1.$$

The quadratic formula now tells us that

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618,$$

where here we have discarded the other root of $p(\lambda)$, $(1-\sqrt{5})/2 \approx -0.618$ since ratios cannot be negative. Observe that the other root is actually

$$1 - \varphi = -\frac{1}{\varphi} = \frac{1-\sqrt{5}}{2}.$$

In 1202, the Italian mathematician Fibonacci described what is now called the *Fibonacci sequence* in his manuscript *Liber Abaci*[1], which appeared earlier in Sanskrit

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[1]Among other things, this is the book that popularized the the numeral system $(0, 1, 2, 3, \cdots, 9)$ among Europeans – he learnt them from Arabic writings of Arab and Persian mathematicians, who themselves learnt it much earlier from writings of Indian mathematicians. This is the reason they are called “Arabic” or “Indo-Arabic” numerals.

poetry by Pingala (∼200 BC). This sequence is defined by

$$F_0 = 0, \qquad F_1 = 1, \tag{1}$$
$$F_n = F_{n-1} + F_{n-2} \qquad (n \geqslant 2). \tag{2}$$

The first few Fibonacci numbers are given by

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots,$$

and this is sequence A000045 in the Online Encyclopedia of Integer Sequences (OEIS). In fact, there are many journals and conferences devoted entirely to the study of this sequence.

An equation like (2) is called a linear recurrence and the Fibonacci numbers are an example of a *linear recurrent sequence* (more on this in §5). Likely you have seen Fibonacci numbers if you ever learned programming; computing the nth Fibonacci number is a basic exercise in recursion. But, perhaps you did not know that there is an explicit formula for the nth Fibonacci number F_n – using the golden ratio, φ! The formula states that

$$F_n = \frac{\varphi^n + (-1)^{n+1}\varphi^{-n}}{\sqrt{5}}. \tag{3}$$

At first glance, this is a surprising connection, even unbelievable – for one thing, the left hand side is clearly an integer while it is completely unclear that the right hand side will be an integer. Hopefully these notes will convince you that (3) is true – maybe even natural.

2. A naive connection

Before getting into the thick of things, let us describe a naive and easy to see connection between φ and F_n. If the limit

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n},$$

exists, then it must be equal to φ. Roughly, this is saying that if n is large, then $F_{n+1} \approx \varphi F_n$. To see this, suppose that the limit exists and equals ρ. Then,

$$\rho = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{F_n + F_{n-1}}{F_n} = \lim_{n\to\infty} \left(1 + \frac{1}{F_n/F_{n-1}}\right) = 1 + \frac{1}{\rho},$$

where we have used the recurrence (2) together with some properties of limits. But this implies that $\rho = \varphi$, since ρ is clearly nonnegative and also a root of $p(\lambda)$.

Of course, this calculation is only valid if we can show that the limit exists, which we do not know *a priori*. This would follow from (3), as for large n, φ^{-n} is negligible in size compared to φ^n, and so $F_n \approx \varphi^n/\sqrt{5}$ and hence

$$\frac{F_{n+1}}{F_n} \approx \frac{\varphi^{n+1}/\sqrt{5}}{\varphi^n/\sqrt{5}} = \varphi.$$

This calculation can be made rigorous using techniques from calculus.

3. Introducing matrices

The key insight that will help us derive (3) is the idea to package consecutive Fibonacci numbers into one unit. That is, we define a vector

$$v_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} \in \mathbb{R}^2.$$

(**Warning:** the index in these notes is slightly off from what I used in class.)

We see that then, the initial value information (1) is given by

$$v_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

What about the recurrence relation, (2)? One has, for $n \geqslant 1$,

$$v_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n \end{bmatrix}.$$

Clearly, this latter depends only on $v_{n-1} = (F_n, F_{n-1})$ and so we find that

$$v_n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = Av_{n-1},$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus, the information in (2) can be represented by the equation $v_n = Av_{n-1}$. But now, it follows simply by repetitive application of the vector recurrence that

$$v_n = Av_{n-1} = A^2 v_{n-2} = A^3 v_{n-3} = \cdots = A^{n-1} v_1 = A^n v_0.$$

Thus, to know F_n, it suffices to compute the action of a large power of A on v_0. This is exactly the situation that merits diagonalization! In class, we did this by writing

$$A = PDP^{-1},$$

where D is diagonal matrix composed of eigenvalues of A and P is a matrix whose corresponding columns are the eigenvectors of A. Then $A^n = PD^nP^{-1}$, and so

$$v_n = PD^nP^{-1}v_0.$$

We know v_0 already, and we can calculate P, D^n and P^{-1} easily by diagonalizing. In these notes we will take a marginally simpler looking approach (though it amounts to the same thing).

4. Diagonalizing the matrix A

4.1. The characteristic polynomial and the eigenvalues. As always in a problem like this, the first step is to compute the characteristic polynomial of A. This is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1.$$

Here we see already a rigorous connection to φ! The characteristic polynomial of A is the same as the polynomial $p(\lambda)$ from the introduction. Thus, we get that A has two distinct eigenvalues, $\lambda_1 = \varphi$ and $\lambda_2 = -\varphi^{-1}$, and hence A is diagonalizable.

4.2. The eigenspace for $\lambda_1 = \varphi$. We need to find the nullspace of

$$A - \lambda_1 I = \begin{bmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{bmatrix}.$$

Doing the elementary row operation $R_1 \leftrightarrow R_2$ followed by $R_2 \to R_2 - (1-\varphi)R_1$, we get

$$\begin{bmatrix} 1 & -\varphi \\ 0 & 0 \end{bmatrix},$$

where we used the fact that

$$1 - (1-\varphi)(-\varphi) = -(\varphi^2 - \varphi - 1) = -p(\varphi) = 0.$$

Thus, a quick calculation tells us that the eigenspace is spanned by $(\varphi, 1)$.

4.3. The eigenspace for $\lambda_2 = -\varphi^{-1}$. A similar calculation will give that the eigenvector for $-\varphi^{-1}$ is $(-\varphi^{-1}, 1)$. Note that there is a symmetry here between φ and $-\varphi^{-1}$ (we replaced φ by $-\varphi^{-1}$ in both the eigenvalue and the eigenvector to get another eigenpair). This is similar to HW 13, Q4 (a) and to the fact that if we take a statement involving the imaginary unit i and replace every occurrence of i with $-i$, this leaves the truth value of the statement unchanged. Here, $\sqrt{5}$ is playing the same role as $i = \sqrt{-1}$; if you took any statement in this set of notes[2] and changed every occurence of $\sqrt{5}$ to $-\sqrt{5}$, the resulting statement will still be true. These are special cases of a general phenomenon that you may learn if you do abstract algebra (MA 450/453/454/553 etc.), which is that certain types of statements cannot distinguish between roots of an irreducible polynomial.

[2] Don't try this at home! I haven't stated this very precisely.

4.4. **Deriving the formula.** We know from a theorem proved in class that if

$$b_1 = \begin{bmatrix} \varphi \\ 1 \end{bmatrix}, \qquad b_2 = \begin{bmatrix} -\varphi^{-1} \\ 1 \end{bmatrix}$$

then $\{b_1, b_2\}$ is linearly independent (since they are eigenvectors of distinct eigenvalues) and hence is a basis for $\mathbb{R}^2$. With some additional effort (do it!), we find that, in fact,

$$v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}} \begin{bmatrix} -\varphi^{-1} \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}(b_1 - b_2).$$

But now,

$$\begin{aligned} v_n = A^n v_0 = A^n \left(\frac{1}{\sqrt{5}}(b_1 - b_2) \right) \\ &= \frac{1}{\sqrt{5}}(A^n b_1 - A^n b_2) \\ &= \frac{1}{\sqrt{5}}(\lambda_1^n b_1 - \lambda_2^n b_2), \end{aligned}$$

where in the penultimate equality we used the fact that A^n is linear while in the ultimate equality, we used the fact that if (λ, b) is an eigenpair for A then (λ^n, b) is an eigenpair for A^n. Plugging in the values, we find that

$$\begin{aligned} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = v_n = A^n v_0 &= \frac{1}{\sqrt{5}} \left(\varphi^n \begin{bmatrix} \varphi \\ 1 \end{bmatrix} - (-\varphi^{-1})^n \begin{bmatrix} -\varphi^{-1} \\ 1 \end{bmatrix} \right) \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} + (-1)^{n+2} \varphi^{-(n+1)} \\ \varphi^n + (-1)^{n+1} \varphi^{-n} \end{bmatrix}. \end{aligned}$$

Comparing the second coordinates of the two sides, we get the desired formula

$$F_n = \frac{\varphi^n + (-1)^{n+1} \varphi^{-n}}{\sqrt{5}}.$$

Amazing!

5. Linear Recurrent Sequences

A similar idea applies generally to linear recurrent sequences which are sequences $(a_n)_{n=0}^{\infty}$ where $d \geqslant 1$ is some fixed number[3], $a_0, a_1, \cdots, a_{d-1}$ are numbers[4] that need to be specified, which satisfy the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_{d-1} a_{n-d+1} + c_d a_{n-d} \qquad (n \geqslant d). \tag{4}$$

[3]called the *degree* of the recurrence
[4]called the *initial values*

We will write[5] $\vec{c} = (c_1, \cdots, c_d)$.

These are called linear since the recurrence (4) shows that a_n depends linearly on the previous d elements in the sequence. The Fibonacci numbers are the basic example with degree $d = 2$ and vector of coefficients $(c_1, c_2) = (1, 1)$.

Example 5.1 (Lucas numbers). The *Lucas numbers* are defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$. Thus, as with the Fibonacci numbers, we have degree $d = 2$ and vector of coefficients $(c_1, c_2) = (1, 1)$, but in this case the initial values are different. The first few elements are

$$2, 1, 3, 4, 7, 11, 18, 29, \cdots,$$

and this is A000032 on OEIS.

Example 5.2 (Tribonacci numbers). If we set $d = 3$, and $(c_1, c_2, c_3) = (1, 1, 1)$, $(a_2, a_1, a_0) = (1, 1, 0)$, we get a sequence whose first few terms are

$$0, 1, 1, 2, 4, 7, 13, 20, \cdots.$$

These are the so-called *Tribonacci numbers*[6], which is A000073 on OEIS.

Example 5.3 (Jacobsthal sequence). If we set $d = 2$, and $(c_1, c_2) = (1, 2)$, $(a_1, a_0) = (1, 0)$, we get a sequence whose first few terms are

$$0, 1, 1, 3, 5, 11, 21, 43, \cdots.$$

This is A001045 on OEIS and is called the *Jacobsthal sequence*.

Example 5.4 (Fibonacci-like exponential sequences). The sequence

$$\Phi = (1, \varphi, \varphi^2, \varphi^3, \varphi^4 \cdots),$$

(i.e., $\Phi_n = \varphi^n$) is the same as the linear recurrent sequence with degree $d = 2$, coefficients $(c_1, c_2) = (1, 1)$, and initial values $(\Phi_1, \Phi_0) = (\varphi, 1)$. This is because when $n \geqslant 2$,

$$\varphi^n = \varphi^{n-2}\varphi^2 = \varphi^{n-2}(\varphi + 1) = \varphi^{n-1} + \varphi^{n-2}.$$

In fact, by replacing every occurrence of $\sqrt{5}$ with $-\sqrt{5}$ (or directly, by a similar calculation), one can see that the sequence

$$\Phi' = (1, -\varphi^{-1}, \varphi^{-2}, -\varphi^{-3}, \varphi^4 \cdots),$$

(i.e., $\Phi'_n = (-\varphi^{-1})^n$) satisfies the same recurrence but with initial data $(\Phi'_1, \Phi'_2) = (-\varphi^{-1}, 1)$ instead.

[5] calling it the *vector of coefficients* or simply *coefficients*

[6] Rest assured, there was no mathematician with the name Tribonacci.

Example 5.5 (General exponential sequences). More generally, for any degree $d \geqslant 1$ and coefficients $\vec{c} = (c_1, c_2, \cdots, c_d)$ giving a recurrence, one can associate with the recurrence an *auxiliary polynomial* (also called characteristic polynomial) defined by

$$p_{\vec{c}}(\lambda) = \lambda^d - c_1\lambda^{d-1} - c_2\lambda^{d-2} - \cdots - c_{d-1}\lambda - c_d.$$

In particular, if λ is any root of this polynomial, then the sequence

$$\Lambda = (1, \lambda, \lambda^2, \lambda^3, \lambda^4, \cdots)$$

satisfies the recurrence $a_n = c_1 a_{n-1} + \cdots c_d a_{n-d}$.

The general idea to solve linear recurrences is to define

$$v_n = \begin{bmatrix} a_{n+d} \\ a_{n+d-1} \\ \vdots \\ a_{n+1} \\ a_n \end{bmatrix} \in \mathbb{R}^d,$$

so that the value of $v_0 = (a_{d-1}, \cdots, a_0)$ encodes the initial values of the sequence, and the recurrence can be written as

$$v_n = Av_{n-1} \qquad (n \geqslant 1),$$

where

$$A = A_{\vec{c}} = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_d \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is a $d \times d$ matrix. Then, if A is diagonalizable, one can find an explicit formula for a_n that depends only on the roots of the characteristic polynomial $p_{\vec{c}}(\lambda)$ and on the initial value data $a_0, \cdots, a_{d-1}$.

6. Underlying Vector Spaces

In this section, we will take a sketchy[7]and much more high-brow approach to our calculations in the hope that it will explain conceptually what is happening. The message we want to convey is that in retrospect it is somewhat natural that matrices can be used here once you realize that the underlying mechanism has a vector space that is behaving nicely.

[7]By this I mean I will not justify every statement – you should think of each statement in this section as having an implicit parenthetical remark saying "check!".

Recall $\mathbb{R}^\infty$, the space of all infinite real sequences[8],

$$\mathbb{R}^\infty = \{(a_n)_{n=0}^\infty : a_n \in \mathbb{R}\}.$$

A cautionary note about notation. When referring to a sequence

$$(a_n)_{n=0}^\infty = (a_0, a_1, a_2, \cdots),$$

we will use a_n to refer to the nth entry in this sequence (this is a real number), the tuple notation (a_n) when we want to refer to the whole sequence as a single object, and a when we want to emphasize the point of view that it is a single object (namely, it is a "point" in the space $\mathbb{R}^\infty$). Sometimes, we will want to take a (finite) sequences of "points" in $\mathbb{R}^\infty$. In this case, we will use superscripts like $b^{(j)}$ to indicate the position to avoid confusion with the entries of the "point" when thought of as a sequence itself.

We discussed a while ago in class[9] that $\mathbb{R}^\infty$ is a vector space under entry-wise addition and scalar multiplication. That is, for $a = (a_n), b = (b_n) \in \mathbb{R}^\infty$ and any $c \in \mathbb{R}$,

$$a + b = (a_n + b_n),$$
$$ca = (ca_n).$$

There is a very natural map acting on $\mathbb{R}^\infty$ called the *shift map* (also called the left-shift operator) which is of fundamental importance in the field of dynamical systems[10]. Precisely $S : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is given by

$$S(a_0, a_1, a_2, a_3, a_4, \cdots) = (a_1, a_2, a_3, a_4, a_5, \cdots).$$

In other words, S deletes the first entry in the sequence, and shifts every other entry to the left. In fact, S is a linear transformation.

Now fix a degree $d \geqslant 1$ and coefficients $\vec{c} = (c_1, \cdots, c_d)$ which determine a recurrence (4). Now, we can define,

$$V_{\vec{c}} = \{a \in \mathbb{R}^\infty : a_n = c_1 a_{n-1} + \cdots c_d a_{n-d} \text{ for } n \geqslant d\}. \tag{5}$$

In other words, $V_{\vec{c}}$ is the set of all sequences that satisfy the satisfying the same recurrence formula (4). It can be shown that $V_{\vec{c}}$ is a subspace of $\mathbb{R}^\infty$. For example, take $F, L \in V_{(1,1)}$ where F is the Fibonacci sequence and L is the Lucas sequence, and define $G = F + L$. The first few elements of this sequence are

$$2, 2, 4, 6, 10, 16, 26, 42, \cdots,$$

[8] In this note, we take the convention that all sequences start at the index 0.

[9] There it was discussed as the canonical example of an *infinite-dimensional vector space*.

[10] Very broadly, dynamical systems is the study of some space under repeated applications of the same map. Each application of the map is to be thought of as evolution under time, so that the asymptotics explain the long-range behaviour of the system after a lot of time. This is useful all over pure and applied mathematics, including in physics, biology, computer science...

which clearly satisfies $G_n = G_{n-1} + G_{n-2}$ for $n \geqslant 2$.

Furthermore, the shift map S restricts to a linear transformation on $V_{\vec{c}}$. That is, if $a \in V_{\vec{c}}$, then $S(a)$ also lies in $V_{\vec{c}}$. Thus, we will abuse notation and write

$$S : V_{\vec{c}} \to V_{\vec{c}}.$$

(If we were not abusing notation, we would call the restriction of S to $V_{\vec{c}}$ something like $S_{\vec{c}}$.)

Note that to specify an element a in $V_{\vec{c}}$ one needs only to specify d real numbers – i.e., the initial values $(a_{d-1}, \cdots, a_0)$. This suggests that $V_{\vec{c}}$ has dimension d, even though $\mathbb{R}^\infty$ is finite dimensional. This can be shown rigorously by finding a basis[11]. Thus, we can summarize this as

Theorem 6.1. *Let $d \geqslant 1$ and $\vec{c} = (c_1, \cdots, c_d)$ and $V_{\vec{c}}$ be as defined in (5). Then $V_{\vec{c}}$ is a finite-dimensional vector subspace of $\mathbb{R}^\infty$, with $\dim V_{\vec{c}} = d$.*

Since $V_{\vec{c}}$ is of dimension d, it must be isomorphic to $\mathbb{R}^d$. In fact, the map $D : V_{\vec{c}} \to \mathbb{R}^d$ given by

$$D(a) = (a_{d-1}, a_{d-2}, \cdots, a_1, a_0),$$

is an explicit isomorphism.

The way all of this theory connects to our earlier calculations is as follows. Define,

$$\mathcal{B} = \{b^{(1)}, \cdots, b^{(d)}\}$$

where each $b^{(j)}$ lies in $\mathbb{R}^\infty$ and is defined by $D(b_j) = e_j$, where

$$e_j = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{R}^d$$

is the standard basis vector having 1 at the jth position and 0s elsewhere. By the discussion above, this set is a basis for V. Further, since S is a linear transformation on $V_{\vec{c}}$, it must have a matrix in the basis $\mathcal{B}$. If one runs through the calculation for this matrix one will find that this is exactly the matrix A from §5! In other words, the matrix $[S]_{\mathcal{B}}$ is the $d \times d$ matrix given by

$$\begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_d \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

To see why this is the case, try to think of what the initial value data of $S(a)$ is when $a \in \mathbb{R}^\infty$ has initial value data $(a_{d-1}, \cdots, a_0)$.

[11]**Hint:** Take $b^{(j)} \in V_{\vec{c}}$ to be the sequence which has the standard basis vector in $e_j \in \mathbb{R}^d$ as its initial value data.

Thus, when we diagonalize A, one can think of that as "diagonalizing" the shift operator S – i.e., we are finding a basis for $V_{\vec{c}}$ in which the matrix of the linear transformation S is diagonal. The elements of this basis are extremely explicit. To be precise, the discussion from Example 5.5 in §5 tells us that if the auxiliary polynommial

$$p_{\vec{c}}(\lambda) = \lambda^d - c_1\lambda^{d-1} - \cdots - c_0$$

has d distinct roots given by $\lambda_j \quad (1 \leqslant j \leqslant d)$, then the sequences $\Lambda^{(j)} \in \mathbb{R}^\infty \quad (1 \leqslant j \leqslant d)$ given by

$$\Lambda^{(j)} = (1, \lambda_j, \lambda_j^2, \lambda_j^3, \lambda_j^4, \cdots)$$

all lie in $V_{\vec{c}}$. It can be shown[12] that these are all linearly independent. Then, if

$$\mathcal{B}' = \{\Lambda^{(1)}, \cdots, \Lambda^{(d)}\},$$

then $\mathcal{B}'$ is the desired basis. Consider the action of S on these elements:

$$\begin{aligned} S(\Lambda^{(j)}) &= S(1, \lambda_j, \lambda_j^2, \lambda_j^3, \lambda_j^4, \cdots) \\ &= (\lambda_j, \lambda_j^2, \lambda_j^3, \lambda_j^4, \lambda_j^5, \cdots) \\ &= \lambda_j(1, \lambda_j, \lambda_j^2, \lambda_j^3, \lambda_j^4, \cdots) \\ &= \lambda_j \Lambda^{(j)}, \end{aligned}$$

and so $\Lambda^{(j)}$ is an "eigenvector"[13] of S with eigenvalue λ_j.

Thus, the reason this technique works is that we happen to have a very concrete description of the eigenvectors of the shift operator (see Problem 8 below).

7. Exercises

These problems are *not* for credit and are only for you to test your understanding. Starred problems will not be tested on the exam.

Problem 1. Using similar techniques to §4, find a formula for the nth entry in the Lucas sequence, the Tribonacci sequence, and the Jacobsthal sequence.

Problem 2. Consider the sequence $a_0 = 0$, $a_1 = 1$ and

$$a_n = 2a_{n-1} - a_{n-2}.$$

Will the techniques from §4 work for this sequence? Why or why not?

[12] Though, not easily – see Problem 7 below.

[13] Eigenvectors of linear transformations $T : V \to V$ are defined to be vectors $v \in V$ satisfying $T(v) = \lambda v$ for some scalar λ. This is in analogy with the matrix case where $V = \mathbb{R}^n$ and T is some matrix operator.

Problem 3. Let $r = (r_n)$ be a sequence satisfying the recurrence with $d = 2$ and $\vec{c} = (-1, -1)$. Pick initial values r_0 and r_1 of your choice and write out the first few elements in this sequence. What do you observe? Will your observation be valid for any initial values? Try to relate this to the characteristic polynomial and the method from §4.

Problem 4. Show that $\{F, L\}$ is a basis for $V_{(1,1)}$. Here F (resp. L) is the Fibonacci (resp. Lucas) sequence.

Problem 5. Show that $\{\Phi, \Phi'\}$ is a basis for $V_{(1,1)}$. Here Φ and Φ' are the Fibonacci-like exponential sequences described in Example 5.4.

Problem 6*. Show that (3) is invariant under the replacement of $\sqrt{5}$ with $-\sqrt{5}$. (**Hint:** you need to change all occurences of $\sqrt{5}$ in the formula, even the ones that are not immediately apparent.)

Problem 7*. Let $d = 3$ and $\vec{c} = (c_1, c_2, c_3)$ be such that

$$\lambda^3 = c_1\lambda^2 + c_2\lambda + c_3$$

has three distinct roots $\lambda_1, \lambda_2, \lambda_3$. Show that $\vec{B}' = \{\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}\}$ is a basis for $V_{\vec{c}}$. [**Hint:** $\mathcal{B}'$ is a basis for $V_{\vec{c}}$ if and only if $D(\mathcal{B}') = \{D(\Lambda^{(1)}, D(\Lambda^{(2)}, D(\Lambda^{(3)}\}$ is a basis for $\mathbb{R}^3$ (why?). Try to relate the question of $D(\mathcal{B}')$ being linearly independent to a Vandermonde matrix (cf. HW 10, Problem 2).]

Problem 8*. Consider the shift map as a linear transformation on the full space $\mathbb{R}^\infty$. Show that for any scalar $\lambda \in \mathbb{R}$, λ is an eigenvalue of S. What is its eigenspace? Funky things can happen here since $\mathbb{R}^\infty$ is an infinite-dimensional space!

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