MATH 142 (SUMMER '21, SESH A2)

ANURAG SAHAY

OFF HRS: M, F 4-5PM ;
BY APPOINTMENT

LECTURES:
5:45 PM – 7:50 PM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

email: anuragsahay@rochester.edu

COURSE PAGE : bit.ly/sahay142

ANNOUNCEMENTS

1. WEBWORK DEADLINES:
   (a) WW 2 → TONIGHT
   (b) WW 3 → TOMORROW
   (c) WW 4 → FRIDAY
2. DEADLINES ARE FLEXIBLE
3. WILL PUT UP PAST EXAMS TONIGHT, SAMPLE MIDTERM TOMORROW.
4. PARTICIPATION GRADES FOR LAST WEEK WILL BE PUT UP TONIGHT
5. WORKSHOP

§ 5.3 FUNDAMENTAL THEOREM OF CALCULUS

RECALL : (THE DEFINITE INTEGRAL)

$$\boxed{\int_a^b f(x)\,dx} = \boxed{\lim_{n \to \infty} \sum_{j=1}^{n} f(x_j^*)\,\Delta x}$$

WHY IS CALC USEFUL THEN?

RHS IS $\boxed{\text{HARD}}$ TO COMPUTE !

LHS ⟿ REPRESENTS AREA / DISTANCE / SOME QUANTITY WE WISH TO COMPUTE

THE INSIGHT OF NEWTON & LIEBNIZ

DIFFERENTIATION $\longleftrightarrow$ INTEGRATION

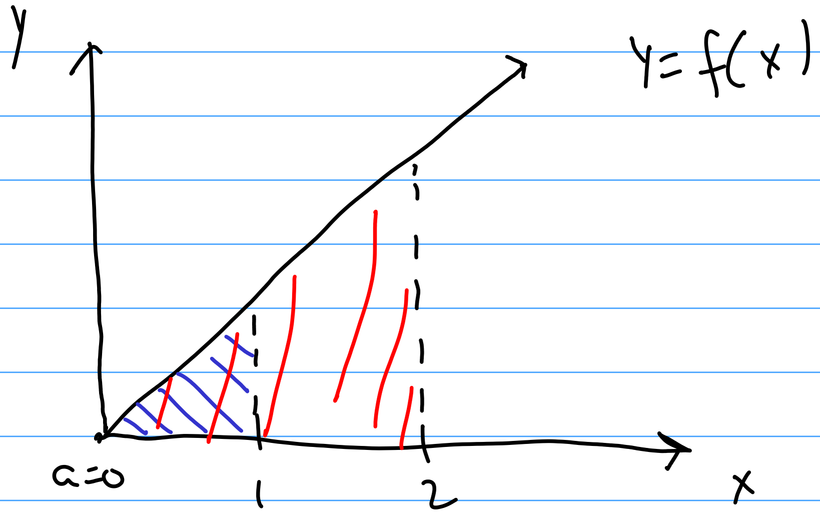
REVERSE OF EACH OTHER!

FUNDAMENTAL THEOREM OF CALCULUS

WE WILL EXPLOIT THIS CONNECTION TO COMPUTE INTEGRALS EASILY.

$a \to$ FIXED REAL #, $f \to$ FIXED FUNCTION

$$g(x) = \int_a^x f(t)\,dt$$



$$g(1) = \int_0^1 f(t)\,dt$$

$$g(2) = \int_0^2 f(t)\,dt$$

$$g(x) = \int_a^x f(t)\,dt$$

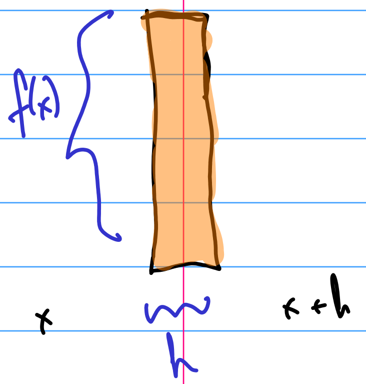
Q. WHAT IS $g'(x)$?

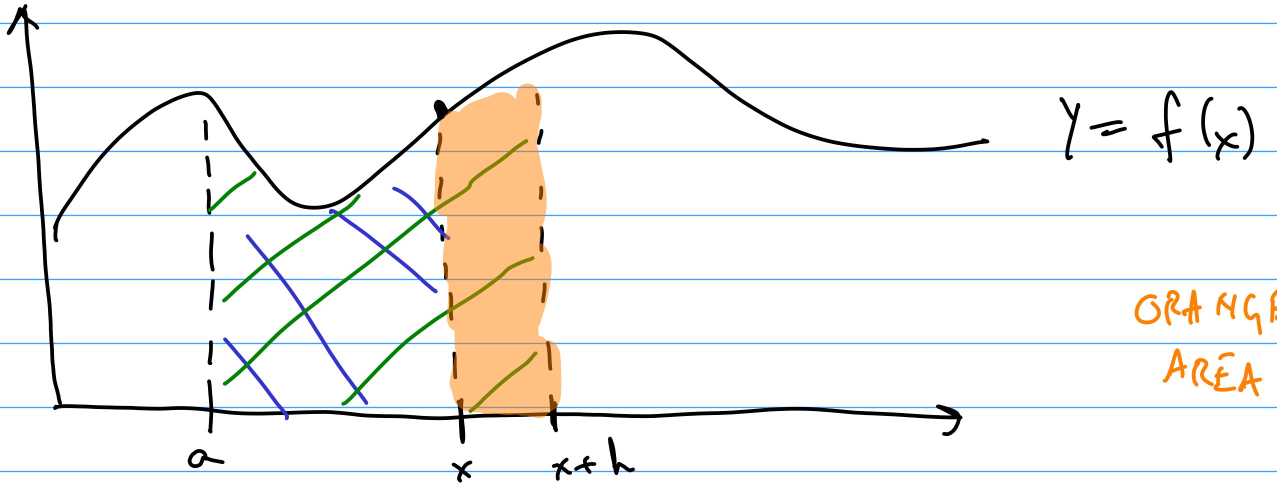
$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

ORANGE AREA $\approx f(x)h$

$h \to 0$

$$g(x) = \int_a^x f(t)\,dt$$





$$g(x+h) = \int_a^{x+h} f(t)\,dt$$

ORANGE AREA $= g(x+h) - g(x)$

$$g(x+h) - g(x) \approx f(x)\,h$$

$$\frac{g(x+h) - g(x)}{h} \approx f(x)$$

$$\Rightarrow \underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}_{g'(x)} = f(x) \qquad \text{[EXPECTATION]}$$

$$\boxed{g'(x) = f(x)}$$

FTC 1

THM: IF f IS CONTINUOUS ON $[a,b]$

& $g(x) = \int_a^x f(t)\,dt$, $a \leq x \leq b$

THEN g IS CONTINUOUS ON $[a,b]$, g IS DIFFERENTIABLE ON (a,b), AND

$$\boxed{g'(x) = f(x)}$$

NOTES

1. PROOF → IN THE BOOK

2. LIEBNIZ NOTATION

$$\underbrace{\frac{d}{dx}\left(\int_a^x f(x)\,dx\right)}_{\frac{d}{dx}(g(x)) \,=\, g'(x)} = f(x)$$

3. $g(x)$ IS AN ANTIDERIVATIVE OF $f(x)$

4. INDEFINITE INTEGRAL $\Longleftrightarrow$ ANTIDERIVATIVE

EXAMPLE 2 Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\, dt$.

$g(x) \int_a^x f(t)\, dt$

$a = 0$

$f(t) = \sqrt{1+t^2}$

$g'(x) = f(x)$

$$g(x) = \int_0^x \sqrt{1+t^2}\, dt$$

$$g'(x) = \sqrt{1+x^2}$$

EXAMPLE 4 Find $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\, dt.$

$$g(x^4) = \int_1^{x^4} \sec t \; dt$$

$$\frac{d}{dx}\left(g(x^4)\right) \qquad u = x^4 \Rightarrow \frac{du}{dx} = 4x^3$$

$$\frac{d}{dx}\left(g(u)\right) = \frac{du}{dx} \cdot g'(u)$$

$$= (4x^3) \sec x^4$$

$$g(x) = \int_a^x f(t)\, dt$$

$f(t) = \sec t$

$a = 1$

$$g(x) = \int_1^x \sec t \, dt$$

$$g'(x) = \sec x$$

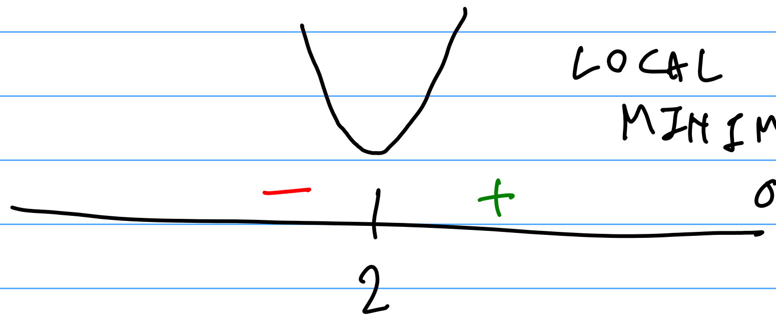
E.g. LET $g(x) = \int_0^x \underbrace{(t-2)}_{f(t)}\,dt$

x = 2 IS A MIN

FIND THE EXTREMUM POINTS OF $g(x)$

POINTS OF MAX/MIN

$g'(x) = x - 2$



LOCAL MINIMUM OF $g(x)$

1st DER.: $g'(x) = 0$

FTC 2

THM: IF f IS CONTINUOUS ON $[a,b]$ AND F IS AN ANTIDERIVATIVE OF f

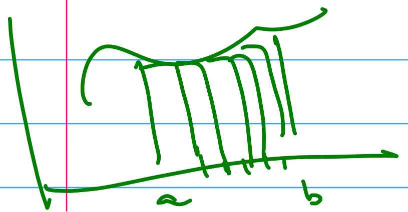
(THAT IS $F' = f$),

THEN

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big]_a^b$$

ONLY END-POINTS

(TAKE A SCREENSHOT)



$$\ln x \Big]_2^3 = \ln 3 - \ln 2$$

NOTES :

(1) RHS DEPENDS ONLY ON END-POINTS.

(2) SUFFICES TO COMPUTE AN ANTIDERIVATIVE

(3) $f = v(t)$

ANTIDER. OF v
= DISPLACEMENT $= s(t)$

$$\underbrace{\int_a^b v(t)\,dt}_{\text{SUM OF INDIVIDUAL MOMENTS}} = \underbrace{s(b) - s(a)}_{\text{HOW MUCH YOU'VE TRAVELLED}}$$

(4) IMPORTANT NOTATION :

$$F(x)\Big]_a^b = F(b) - F(a)$$

EXAMPLE 5 Evaluate the integral $\boxed{\int_1^3 e^x \, dx.}$

? ? $$\sum_{i=1}^{n} e^{1+\frac{2i}{n}} \cdot \frac{1}{n}$$? ? ?

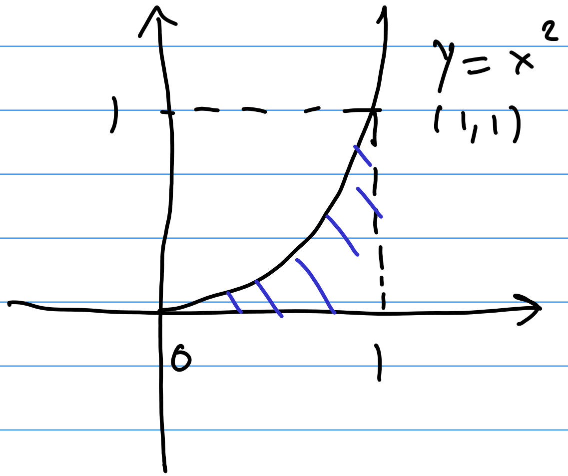
$$\int_a^b f(x)\, dx = F(x)\Big]_a^b$$

$a=1$, $b=3$, $f(x) = e^x$ $\quad F(x)$?

$$F(x) = e^x$$

$$\int_1^3 f(x)\,dx = e^x\Big]_1^3 = e^3 - e^1 = e^3 - e$$

EXAMPLE 6 Find the area under the parabola $y = x^2$ from 0 to 1.



$$\text{AREA} = \int_0^1 x^2\,dx$$

$$(\text{FTC } 2) = \left.\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3}$$

$$= 1/3$$

$$\int_a^b f(x)\,dx$$

$a = 0$

$b = 1$

$f(x) = x^2$

$$F(x) = \frac{x^3}{3}$$

$$x^n \xrightarrow[\text{DERIVATIVES}]{\text{ANTI-}} \frac{x^{n+1}}{n+1}$$

Pf $\boxed{\text{FTC1} \Rightarrow \text{FTC2}}$

TO SHOW

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$g(x) = \int_a^x f(x)\,dx \xRightarrow[\text{FTC1}]{} g'(x) = f(x)$$

$\Rightarrow$ g IS AN ANTIDER. OF f.

$$g(x) = \int_a^x f(x)\,dx$$

$$F(x) = g(x) + C$$

$$\Rightarrow F(b) - F(a) = \big(g(b) + C\big) - \big(g(a) + C\big)$$

$$= g(b) - g(a) = g(b) = \int_a^b f(x)\,dx$$

$$g(a) = \int_a^a f(x)\,dx = 0$$

$\blacksquare$

BREAKOUT ROOMS

EXAMPLE 7 Evaluate $\int_3^6 \frac{dx}{x}$.

EXAMPLE 9 What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Bigg]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

RETURN AT

7:05 PM EST

REVERSE

DIFF $\longleftrightarrow$ INT

FTC 1 : $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

FTC 2 : $\int_a^b F'(x)\,dx = F(b) - F(a)$

FTC 1 : $f(x) \xrightarrow{\text{INT}} \int_a^x f(t)\,dt \xrightarrow{\text{DIFF}} \frac{d}{dx}\int_a^x f(t)\,dt \rightarrow f(x)$

FTC 2 : $F(x) \xrightarrow{\text{DIFF}} F'(x) \xrightarrow{\text{INT}} \int_a^b F'(x)\,dx \rightarrow F(b) - F(a)$

§ 5.4 INDEFINITE INTEGRALS AND THE NET CHANGE THEOREM

NOTATION : $F \rightarrow$ ANTIDERIVATIVE / INDEFINITE INTEGRAL OF f

$\int_a^b f(x)\,dx \rightarrow$ DEFINITE

$$F(x) = \int f(x)\,dx$$

(NO LIMITS OF INTEGRATION)

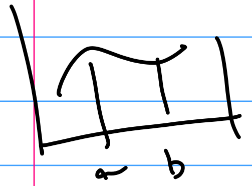
$$\Updownarrow$$

$$F'(x) = f(x)$$

e.g. $\int x^2\,dx = \frac{x^3}{3} + C \iff \frac{d}{dx}\left[x^3/3 + C\right] = x^2$

N.B. DEFINITE VERSUS INDEFINITE

$\int_a^b f(x)\,dx$ $\int f(x)\,dx$



REAL NUMBER FUNCTION

$F = \int f\,dx \Rightarrow F' = f$

GEOMETRIC $\longleftrightarrow$ ALGEBRAIC

REASON FOR NOTATION: FTC2

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$= F(x)\Big]_a^b$$

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$$

RULES

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

(POWER)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

N.B. HOW TO KNOW YOU FOUND THE RIGHT INDEFINITE INTEGRAL?

A: DIFFENTIATE THE ANSWER

EXAMPLE 1 Find the general indefinite integral $\longleftrightarrow$ ANTIDERIVATIVE

$$\int (10x^4 - 2\sec^2 x)\,dx = 2x^5 - 2\tan x + C$$

$$\int (10x^4 - 2\sec^2 x)\,dx = \left(\int 10x^4\,dx\right) - \left(\int 2\sec^2 x\,dx\right)$$

$$\frac{10x^5}{5} - 2\tan x$$

EXAMPLE 5 Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$.

$(2t^2 + t^2\sqrt{t} - 1)\, t^{-2}$

$= (2t^2)\cdot t^{-2}$

$$\frac{2t^2 + t^2\sqrt{t} - 1}{t^2} = \frac{2t^2}{t^2} + \frac{t^2\sqrt{t}}{t^2} - \frac{1}{t^2}$$

$$= 2 + \sqrt{t} - t^{-2}$$

$$\int_1^9 \left(2 + \sqrt{t} - t^{-2}\right) dt = \left[2t + \frac{t^{3/2}}{3/2} - \left(\frac{t^{-1}}{-1} \right) \right]_1^9$$

$$= \left(2(9) + \frac{9^{3/2}}{3/2} + 9^{-1} \right) - \left(2 + \frac{1^{3/2}}{3/2} - 1^{-1} \right)$$

NET CHANGE THEOREM

(A.K.A. FTC2 FOR PROFESSIONALS)

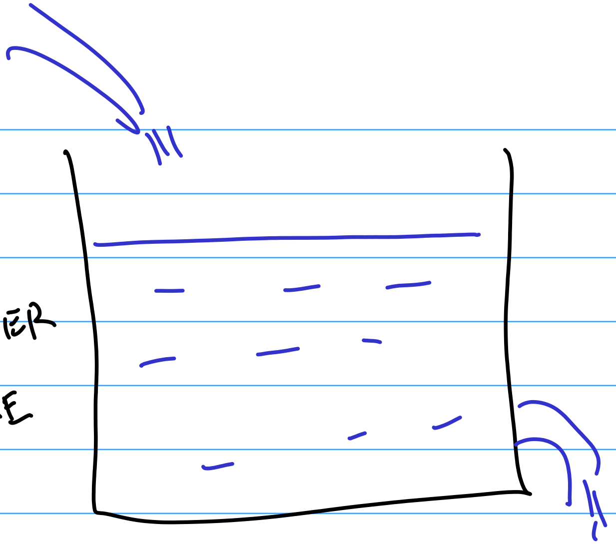
THM: "THE INTEGRAL OF A RATE OF CHANGE IS THE **NET** CHANGE"

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

* APPLIES TO ANY RATE OF CHANGE

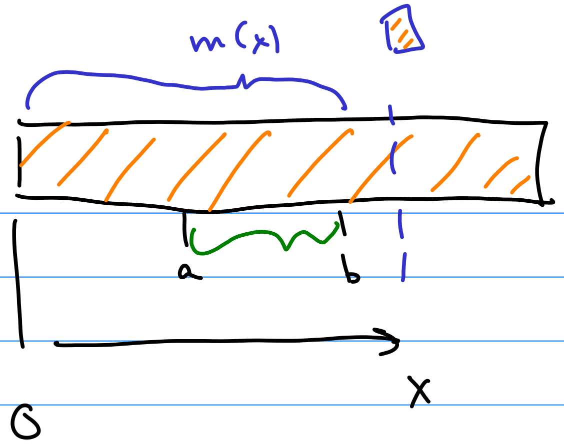
e.g.

$V(t)$ ↳ VOLUME OF WATER AT TIME



$$\underbrace{\int_{t_1}^{t_2} V'(t)\,dt}_{\text{INTEGRAL OF RATE OF CHANGE}} = \underbrace{V(t_2) - V(t_1)}_{\text{NET CHANGE}}$$

eg.



m(x) = MASS BETWEEN 0 TO x

$\rho(x) = \frac{dm}{dx}$ → DENSITY AT x

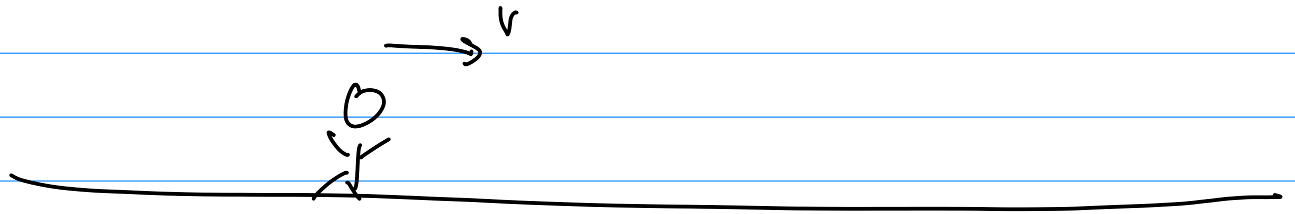
INTEGRAL OF RATE OF CHANGE ← $\int_a^b \rho(x)\,dx = m(b) - m(a)$ → NET CHANGE eg.

n = POPULATION OF GEESE IN A POND

$\frac{dn}{dt} \to$ RATE OF CHANGE

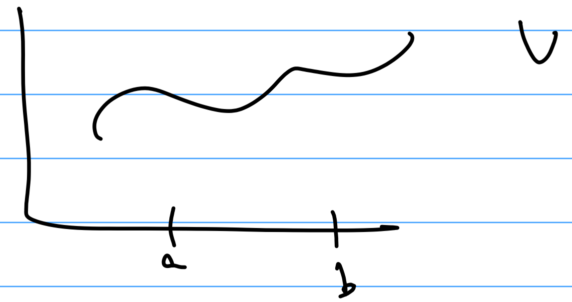
$$\int_{t_1}^{t_2} \left(\frac{dn}{dt}\right) dt = n(t_2) - n(t_1)$$

DISPLACEMENT & DISTANCE



$t = a$ $\qquad\qquad$ $t = b$

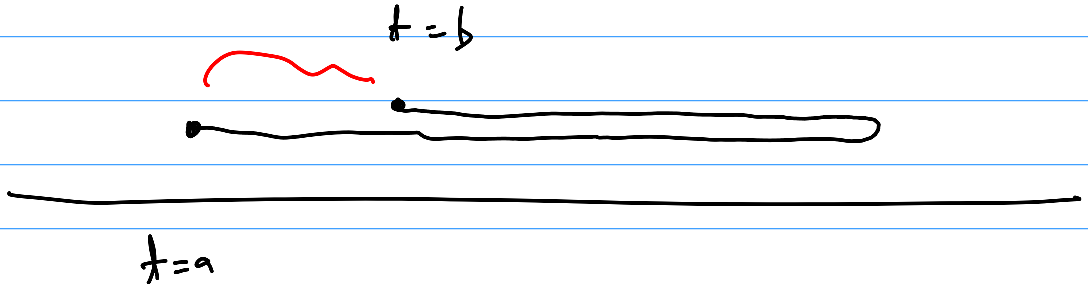
$v(t)$



$$\int_a^b v(t)\,dt = s(b) - s(a)$$

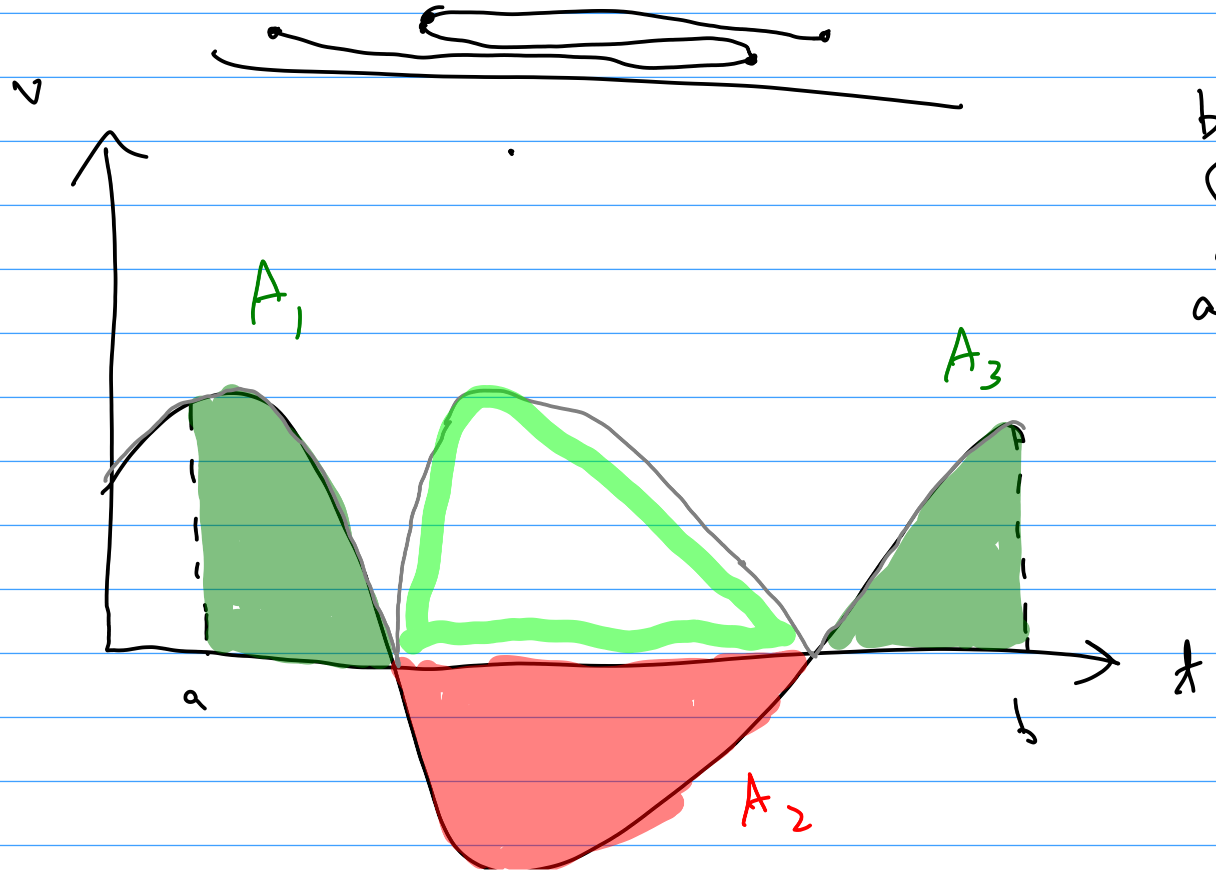
$s \rightarrow$ DISPLACEMENT

$$\frac{ds}{dt} = v$$



$$s(b) - s(a)$$
$$= \int_a^b v(t)\,dt$$

DISTANCE TRAVELLED $= \int_a^b |v(t)|\,dt$



$$\int_a^b v(t)\,dt = A_1 - A_2 + A_3$$

$$\int_a^b |v(t)|\,dt = A_1 + A_2 + A_3$$

EXAMPLE 6 A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leqslant t \leqslant 4$.

(b) Find the distance traveled during this time period.