MATH 201 HW 6

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The probability density function of an exponential random variable with parameter λ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

A random variable X with this pdf is such that $E[X] = 1/\lambda$. If X represents the lifetime of the 60 watt bulb with an expected lifetime of 200 days, it has a pdf

$$f_X(x) = \begin{cases} \frac{e^{-x/200}}{200} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

For Y representing the lifetime of the 100 watt bulb with an expected lifetime of 100 days we get

$$f_Y(y) = \begin{cases} \frac{e^{-y/100}}{100} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

To calculate the cdf, we find (for $a \geq 0$)

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} p_X(x)dx = \frac{1}{200}\int_0^a e^{-x/200}dx = -e^{-x/200}\Big|_0^a = 1 - e^{-a/200}$$

Since $P(X \leq 0) = 0$, we get the cdf

$$F_X(a) = \begin{cases} 1 - e^{-a/200} & a \geq 0 \\ 0 & a < 0 \end{cases}$$

Similarly, for $b \geq 0$

$$F_Y(b) = P(Y \leq b) = \int_{-\infty}^{b} p_Y(y)dy = \frac{1}{100}\int_0^b e^{-y/100}dy = -e^{-y/100}\Big|_0^b = 1 - e^{-b/100}$$

Since $P(Y \leq 0) = 0$, we get the cdf

$$F_Y(b) = \begin{cases} 1 - e^{-b/100} & b \geq 0 \\ 0 & a < 0 \end{cases}$$

From the above computations,

$$P(X < Y) = \int_0^{\infty} \frac{e^{-x/200}}{200}(1 - (1 - e^{-x/100}))dx = \frac{1}{200}\int_0^{\infty} e^{-3x/200}dx = -\frac{1}{3}e^{-3x/200}\Big|_0^{\infty} = \frac{1}{3}$$

If they both run for more than 10 days, the probability remains the same. Recall the memory-less property of the exponential distribution:

$$P(X > a + x | X > a) = P(X > x)$$

The same situation holds here:

$$P(Y > X|X, Y > 10) = \frac{P(\{Y > X\} \cap \{X, Y > 10\})}{P(X, Y > 10)} = \frac{P(\{Y > X\} \cap \{X > 10\})}{P(X > 10)P(Y > 10)}$$

The numerator is the integral

$$\int_{10}^{\infty} f_X(x)(1 - F_Y(x))dx = -\frac{1}{3}e^{-3x/100}\Big|_{10}^{\infty} = \frac{1}{3}e^{-30/200}$$

The denominator is the integral

$$\int_{10}^{\infty}\int_{10}^{\infty} f_X(x)f_Y(y)dxdy = (1 - F_X(10))(1 - F_Y(10)) = e^{-10/200}e^{-10/100} = e^{-30/200}$$

Therefore

$$P(Y > X|X, Y > 10) = \frac{\frac{1}{3}e^{-30/200}}{e^{-30/200}} = \frac{1}{3}$$

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If X and Y are independent, then

$$P(Y > X) = \sum_{k=1}^{\infty}(1-p)^{k-1}p\sum_{j=k+1}^{\infty}(1-r)^{j-1}r = \sum_{k=1}^{\infty}(1-p)^{k-1}p(1-r)^k r\sum_{j=1}^{\infty}(1-r)^{j-1}$$

This simplifies to

$$\sum_{k=1}^{\infty}(1-p)^{k-1}p(1-r)^k r\frac{1}{r} = \sum_{k=1}^{\infty}(1-p)^{k-1}p(1-r)^k = p(1-r)\sum_{k=1}^{\infty}((1-p)(1-r))^{k-1}$$

This simplifies to

$$\frac{p(1-r)}{1-(1-p)(1-r)}$$