Moments of the Hurwitz zeta function and related topics

by

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Dedicated to Dr. Shobha Madan and Dr. Suchitra Mathur; the former for her mentorship, and the latter for her friendship.

Table of Contents

Biographical Sketch

The author of this thesis was born in Jamshedpur, India. He started an undergraduate degree at the Indian Institute of Technology in July, 2011 and graduated in May, 2016 with a dual degree (B. S. and M. S.) in "Mathematics and Scientific Computing". He received IIT Kanpur's Ratan Swarup memorial prize for best all round graduating undergraduate in 2016. Following that, he worked at the New Delhi office of the Boston Consulting Group from September, 2016 to September, 2017. After a year of existential contemplation and not much else, he joined the Mathematics PhD program at the University of Rochester in August, 2018, where he worked as a teaching assistant and occasionally as an instructor. His doctoral studies were supervised by Prof. Steven M. Gonek.

The following articles by the author were published in the indicated journals:

1. Alex McDonald, Brian McDonald, Jonathan Passant, and Anurag Sahay, *Distinct distances from points on a circle to a generic set.* **Integers**, 21: Paper No. A55, 2021. (arXiv:2005.02951)

2. Winston Heap, Anurag Sahay, and Trevor D. Wooley, *A paucity problem associated with a shifted integer analogue of the divisor function.* **J. Number Theory**, 242:660–668, 2023. (doi:10.1016/j.jnt.2022.05.006, arXiv:2108.00287)

3. Anurag Sahay, *Moments of the Hurwitz zeta function on the critical line.* **Math. Proc. Cambridge Phil. Soc.**, 174(3):631–661, 2023. (doi:10.1017/S0305004122000457, arxiv:2103.13542)

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Abstract

In this thesis, we explore the value distribution of the Hurwitz zeta function, $\zeta(s,\alpha)$, specifically its moments on the critical line $s = 1/2 + it$, and some related problems. When α is rational, this leads naturally to the study of moments of products of Dirichlet L-functions on the critical line which are studied following the approach of Gonek–Hughes–Keating [31] and Heap [38]. When α is irrational, this connects to an interesting Diophantine problem which exhibits a paucity phenomenon. On the basis of these considerations and others, we conjecture that

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^{2k}\, dt \sim c_k(\alpha) T (\log T)^{k^2},$$

for an explicit constant $c_k(\alpha)$ when α is rational, while

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^{2k} \sim k! T (\log T)^{k},$$

when α is algebraic of degree $d \geqslant k$ or α is transcendental but for possibly an exceptional set of null Lebesgue measure. Some of this includes joint work with Winston Heap, and joint work with Winston Heap and Trevor Wooley.

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This work was supervised by Professor Steven M. Gonek (adviser) of the Department of Mathematics. The dissertation committee also included Professor Dinesh S. Thakur (internal member) of the Department of Mathematics, Professor Sarada G. Rajeev (external member) of the Department of Physics and Astronomy, and Professor Paul Audi (chair) of the Department of Philosophy.

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Theorem 4.1.1 is joint work with Winston Heap and Trevor Wooley. The rest of Chapter 4 is joint work with Winston Heap.

1 Introduction

The central object of this thesis is the Hurwitz zeta function, $\zeta(s,\alpha)$. For a shift parameter[1] $0 < \alpha \leqslant 1$, $\zeta(s,\alpha)$ is defined by the infinite sum

$$\zeta(s,\alpha) = \sum_{n \geqslant 0} \frac{1}{(n+\alpha)^s},$$

for $s = \sigma + it \in \mathbb{C}$, $\sigma > 1$, where the sum here runs over all non-negative integers, including 0. Setting $\alpha = 1$, one finds that

$$\zeta(s,1) = \zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$$

is the usual zeta function of Riemann. One can also check that

$$\begin{aligned}\zeta(s,\tfrac{1}{2}) &= \frac{1}{(1/2)^s} + \frac{1}{(3/2)^s} + \frac{1}{(5/2)^s} \cdots \\ &= 2^s\Big(1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots\Big) \\ &= 2^s\Big(1 - \frac{1}{2^s}\Big)\zeta(s) = (2^s - 1)\zeta(s).\end{aligned}$$

However, when $\alpha \neq 1/2, 1$, there are no simple relationships between $\zeta(s,\alpha)$ and $\zeta(s)$.

Both zeta functions satisfy many similar properties:

[1] In principle, one may take $\alpha \in \mathbb{R} \setminus \{0,-1,-2,\cdots\}$, but it is typical to restrict to this interval due to the simple relationship $\zeta(s,\alpha) - \zeta(s,\alpha+1) = \alpha^{-s}$.

- They both converge absolutely in $\sigma > 1$, and uniformly on $\sigma \geqslant \sigma_0 > 1$, thereby defining a holomorphic function on $\sigma > 1$.

- They both extend to meromorphic functions on $\mathbb{C}$ with a simple pole at $s = 1$, with residue 1.

- They both have "trivial" zeros on the negative real line, but are zero-free [2] in the region $\sigma \geqslant 1 + \alpha$ [80].

- They both satisfy a "functional equation" (see §1.1).

Despite these similarities, however, there are considerable differences between $\zeta(s)$ and $\zeta(s, \alpha)$ when $\alpha \neq 1/2, 1$:

- The Riemann zeta function has an Euler product,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

  for $\sigma > 1$. The Hurwitz zeta function $\zeta(s, \alpha)$ does not.

- For any $\delta > 0$, $\zeta(s, \alpha)$ has infinitely many zeroes with $1 < \sigma < 1 + \delta$. This is due to Davenport and Heilbronn [24] for rational and transcendental shifts; and due to Cassels [18] for algebraic irrational shifts). In particular, the region $1 < \sigma < 1 + \alpha$ is not zero-free, despite being in the domain of absolute convergence!

- For σ_1, σ_2 with $1/2 < \sigma_1 < \sigma_2 < 1$, there are infinitely many zeroes of $\zeta(s, \alpha)$ in the strip $\sigma_1 < \sigma < \sigma_2$, due to Voronin [84] for rational shifts and due to Gonek [32] for transcendental shifts. In fact, one can show that there are $\asymp_{\sigma_1, \sigma_2, \alpha} T$ many such zeroes up to height T. This is likely also

[2] The existence of an $A = A(\alpha)$ such that $\zeta(s, \alpha) \neq 0$ for $\sigma > A$ is straightforward, since as $\Re(s) \to \infty$, the first term in $\zeta(s, \alpha)$ decays exponentially slower than the other terms, giving $\zeta(s, \alpha) = \alpha^{-s}(1 + o(1))$.

true for algebraic irrationals, but this question appears to be open[3]. This is in contrast with the Riemann zeta function, both in terms of its expected behaviour[4] and in terms of its known behaviour[5].

The astute reader will have noticed that there appears to be a theme of trifurcating depending on whether α is rational, transcendental, or algebraic irrational. This is not a coincidence; in contexts where some result about $\zeta(s,\alpha)$ is known for all three classes, the flavours of and difficulties in the proof depend heavily on where α falls in this trichotomy. The most interesting case from the perspective of arithmetic is when $\alpha \in \mathbb{Q}$. In this case, we write $\alpha = a/q$, with $(a,q)=1$, and further, we can assume that $q \geqslant 3$. One has that

$$\zeta(s, \tfrac{a}{q}) = \sum_{n\geqslant 0} \frac{1}{\left(n+\frac{a}{q}\right)^s} = q^s \sum_{n\geqslant 0} \frac{1}{(qn+a)^s} = q^s \sum_{\substack{m\geqslant a \\ m\equiv a \pmod q}} \frac{1}{m^s}.$$

Since $(a,q)=1$, one can now use the orthogonality of Dirichlet characters to write

$$\mathbb{1}_{m\equiv a \pmod q} = \frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a)\chi(m),$$

where here and throughout, $\mathbb{1}_P$ represents the indicator function of whether the predicate P is true, and where sums over χ are taken over all Dirichlet characters modulo q. Substituting this above, and interchanging the order of summation, one obtains

$$\frac{q^s}{\varphi(q)} \sum_{\chi} \overline{\chi}(a) \sum_{m\geqslant 1} \frac{\chi(m)}{m^s},$$

whence

$$\zeta(s, \tfrac{a}{q}) = \frac{q^s}{\varphi(q)} \sum_{\chi} \overline{\chi}(a) L(s,\chi), \tag{1.1}$$

[3]The preprint [62] has recently appeared, claiming to settle this problem for infinitely many algebraic irrationals α.

[4]viz., the Riemann hypothesis – that there are no such zeroes

[5]since, due to zero density estimates, we know that there are $\ll_{\sigma_1,\sigma_2} T^{1-\delta}$ zeroes of $\zeta(s)$ with $\Re(s)$ in the same strip, where $\delta = \delta(\sigma_1,\sigma_2) > 0$ may depend on the strip

where $L(s,\chi)$ is the Dirichlet L-function associated with χ. By analytic continuation, (1.1) continues to hold wherever both sides are holomorphic, i.e. for $s \in \mathbb{C} \setminus \{1\}$. Thus, at least in principle, any question about the Hurwitz zeta function for rational parameters can be converted into a question about linear combinations of Dirichlet L-functions.

For $\alpha \notin \mathbb{Q}$, no relationship similar to (1.1) is known. For algebraic α, number theoretic aspects of the number field $K = \mathbb{Q}(\alpha)$ may play a role[6], but note that algebraic integers of the form $n+\alpha$ typically constitute a very sparse subset of the ring of integers $\mathfrak{O}_K$, and so direct appeals to the arithmetic of K are of limited value. When α is transcendental, by contrast, the techniques have an analytic and probablistic flavor, as the harmonics

$$\{(n+\alpha)^{-it} : n \geqslant 0\},$$

behave essentially like independent random variables; this is because the set

$$\{-\log(n+\alpha) : n \geqslant 0\}, \tag{1.2}$$

is clearly linearly independent over $\mathbb{Q}$, and hence one may apply the Kronecker–Weyl theorem.

The linear independence of (1.2) over $\mathbb{Q}$ is of central importance in the universality of the Hurwitz zeta function, and is the subject of a question raised by Drungilas and Dubickas [26] – they essentially ask if (1.2) is always linearly dependent when α is an algebraic number. The reader is referred to a nice manuscript by Andersson [2] which discusses this question, its connection the universality of $\zeta(s,\alpha)$, and gives evidence that the answer to the question is yes by proving that it follows from a conjecture of Martin [60] on smooth values of polynomials.

[6] For example, in [18], Cassels uses properties of $\mathbb{Q}(\alpha)$ to prove that $> 50\%$ of (1.2) is linearly independent when α is an algebraic irrational; this suffices for his purposes. See [87] for an extension of this result.

In this thesis, we will explore the moments of the Hurwitz zeta function on the critical line. We will also investigate some ancillary Diophantine equations that arise in the study of these moments. To define the basic object of study, let $k \geqslant 0$ be a fixed real number (usually an integer), and let T be large. Then, in analogy with the moments of the Riemann zeta function (see (1.6) and other discussion in §1.2), we define the moments of the Hurwitz zeta function[7] by

$$M_k(T;\alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2}+it,\alpha)|^{2k}\, dt, \tag{1.3}$$

so that $M_k(T;1) = M_k(T)$. The goal of this thesis is to convince the reader of the truth of the following two conjectures:

Conjecture 1.0.1 (S., 2023)**.** *Let $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$ be a fixed rational. Then, we have*

$$M_k(T;\alpha) \sim c_k(\alpha) T (\log T)^{k^2}$$

as $T \to \infty$, where $c_k(\alpha)$ is a constant depending only on k and α. In particular, if $\alpha = a/q$ with $(a,q)=1$, then

$$c_k(\alpha) = c_k \frac{q^k}{\varphi(q)^{2k-1}} \prod_{p \mid q} \left\{ \sum_{m=0}^{\infty} \binom{m+k-1}{k-1}^2 p^{-m} \right\}^{-1}, \tag{1.4}$$

where $c_k = c_k(1)$ is the usual proportionality constant for moments of $\zeta(s)$.

Conjecture 1.0.2 (Heap–S., 2023+)**.** *Let $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$ be a fixed irrational. Then for algebraic α of degree $d \geqslant k$ and almost all[8] transcendental α we have*

$$M_k(T;\alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2}+it,\alpha)|^{2k} dt \sim k!\, T(\log T)^k$$

as $T \to \infty$.

[7] Here, we are interested almost exclusively in the t-aspect. One may also integrate over the α-aspect – see [28, Chapter 1.5] and references therein – but we shall refrain from discussing this further.

[8] in the sense that the excecptional set is of null Lebesgue measure

Conjecture 1.0.1 is the subject of the author's published work [75], while Conjecture 1.0.2 is from ongoing work [41] joint with Winston Heap.

In the rest of this introduction, we first provide an exposition of the salient properties of the Hurwitz zeta function and then review moments in the classical settings of the Riemann zeta function and its connection to random matrix theory. In Chapter 2 and Chapter 3, we present the author's work [75] by specializing to $\alpha = \frac{a}{q} \in \mathbb{Q}$; in the former chapter, we study moments of products of Dirichlet L-functions, which arise naturally due to (1.1), while in the latter chapter we use these to study the moments of $\zeta(s, \frac{a}{q})$. In Chapter 4, we justify Conjecture 1.0.2 by first considering the pseudomoments of the Hurwitz zeta function for $\alpha \notin \mathbb{Q}$ – this is a simplified model for the moments in which many interesting features of the Hurwitz zeta function already display themselves. This includes some unpublished joint work with Winston Heap, and connects to our ongoing work [41] on the fourth moment of the Hurwitz zeta function. Finally, in the same chapter, we also discuss an interesting Diophantine problem arising from these considerations which was the subject of joint work of Winston Heap, the author, and Trevor Wooley in [42]. Independently, this latter problem had been previously investigated by Bourgain, Garaev, Konyagin, and Shparlinski [12] in a different context; our results are essentially equivalent, and the proofs very similar.

1.0.1 Notation

We will use the standard asymptotic notations $\ll, \gg, \asymp, \sim, O(\cdot), o(\cdot)$ as an analytic number theorist would without further comment. If the implicit constant or the rate of convergence may depend on some parameters, we may specify these parameters using subscripts. Thus, $A \ll_\delta B$ means that $A \leqslant C_\delta B$ for some C_δ which may depend on δ but is uniform in the other parameters.

We also adopt the ϵ-notation: each occurence of ϵ represents a sufficiently small positive real number which may vary from occurrence to occurrence, potentially even in the same line.

Finally, we collect here some notation for the reader's convenience – these will be reiterated when they are first used:

- We use the analytic number theorist's shorthand $e(t) := e^{2\pi i t}$ for additive characters.

- If P is a predicate, then $\mathbb{1}_P$ is the indicator of the predicate, i.e.,

$$\mathbb{1}_P = \begin{cases} 1 \text{ if } P \text{ is true,} \\ 0 \text{ otherwise.} \end{cases}$$

- Unless stated otherwise, χ and ν always represent Dirichlet characters modulo q, and sums of the shape

$$\sum_{\chi}, \qquad \sum_{\chi,\nu},$$

  are taken over all such Dirichlet characters.

- $d_k(n)$ represents the k-fold divisor function, i.e., the coefficient of n^{-s} in the Dirichlet series of $\zeta(s)^k$.

- $G(k)$ is the Barnes G-function.

- The bolded letter $\boldsymbol{\ell}$ represents a function on the set of Dirichlet characters mod q, $\mathcal{D}(q)$ taking positive integer values. The integer $\boldsymbol{\ell}(\chi)$ will be denoted as ℓ_χ. Furthermore, we define

$$|\boldsymbol{\ell}| = \sum_{\chi} \ell_\chi, \qquad \lambda(\boldsymbol{\ell}) = \sum_{\chi} \ell_\chi^2, \qquad \mathcal{L}^{\boldsymbol{\ell}}(s) = \prod_{\chi} L(s,\chi)^{\ell_\chi}.$$

- We use $d_{\boldsymbol{\ell}}(n)$ for the coefficient of n^{-s} in the Dirichlet series for $\mathcal{L}^{\boldsymbol{\ell}}(s)$.

- $\mathsf{GRH}(q)$ denotes the statement that the Generalized Riemann Hypothesis holds for $L(s,\chi)$ for every character χ modulo q.
- $\mathsf{Sp}(q,k)$ denotes the statement that the mean-squares of $\mathcal{L}^{\boldsymbol{\ell}}(s)$ on the critical line for all $|\boldsymbol{\ell}| = k$ satisfy Conjecture 2.1.2 and Conjecture 2.1.4.

1.1 Historical remarks and functional equation

According to Davenport [23, Chapter 9], $\zeta(s,\alpha)$ was first introduced by Adolf Hurwitz in 1882 as part of his proof of the functional equation for quadratic Dirichlet L-functions. While in many modern accounts, it is typical to prove the functional equation for $L(s,\chi)$ by using the Poisson summation formula twisted by χ (or equivalently, by using the modularity of the theta function associated with χ), Hurwitz's original proof proceeded via the "functional equation" of the Hurwitz zeta function. Hurwitz's idea applies also to complex characters, but, as Davenport mentions, he was interested primarily in the theory of quadratic forms, and hence did not treat higher order characters. The proof proceeds by inverting (1.1) to get that

$$L(s,\chi) = q^{-s} \sum_{\substack{1\leqslant a\leqslant q\\ (a,q)=1}} \chi(a)\zeta(s,\tfrac{a}{q}),$$

and then applying the functional equation for $\zeta(s,\frac{a}{q})$.

To state the functional equation for $\zeta(s,\alpha)$, one needs to introduce the additively twisted zeta function

$$P(s,\alpha) = \sum_{n\geqslant 1} \frac{e(n\alpha)}{n^s},$$

defined, for $\alpha \in \mathbb{R}$, initially in the region of absolute convergence $\Re(s) > 1$, and then extended to $s \in \mathbb{C}$ by analytic continuation except for a possible pole[9] at

[9] which occurs if and only if $\alpha \in \mathbb{Z}$

$s = 1$. Here and throughout, $e(t) = e^{2\pi it}$. This function is sometimes called the *periodic zeta function* or the *periodized zeta function* in the literature. The functional equation then states,

$$\zeta(1-s,\alpha) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi is/2}P(s,\alpha) + e^{\pi is/2}P(s,-\alpha)\right), \tag{1.5}$$

where $\Gamma(s)$ is the gamma function. A grouchy reader may object that since this identity does not relate the values of $\zeta(s,\alpha)$ to itself, it should not be called a functional equation. This hypothetical reader may rest assured that (1.5) is a special case of a genuine functional equation for the Lerch zeta function (see, for example, [5] or [59, Chapter 2]).

An accessible account of a proof of the functional equation of the Hurwitz zeta function in the form (1.5) may be found in [56]. A proof along classical lines may be found in [6, §12.7]; Apostol calls it "Hurwitz's formula"[10], and uses it, presumably following Hurwitz, to prove the functional equations of $\zeta(s)$ and $L(s,\chi)$.

Since $\zeta(s) = \zeta(s,1) = P(s,\pm 1)$, setting $\alpha = 1$ in (1.5) yields

$$\zeta(1-s) = \frac{\Gamma(s)\cos(\frac{\pi s}{2})}{2^{s-1}\pi^s}\zeta(s),$$

which is a rearrangement of the usual functional equation of $\zeta(s)$.

As a final remark to close out this section, note that when s is high up on the critical line, (1.5) gives

$$\zeta(\tfrac{1}{2}+it,\alpha) = \mathscr{X}(\tfrac{1}{2}+it)P(\tfrac{1}{2}-it,-\alpha)(1+o(1)),$$

where $\mathscr{X}$ is the usual χ-factor in the functional equation $\zeta(s) = \mathscr{X}(s)\zeta(1-s)$. This is because of the rapid decay of the exponential factor in $e^{-\pi is/2}P(s,\alpha)$ when $\Im(s) \to -\infty$. This suggests that, perhaps, $\zeta(s,\alpha)$ and $P(s,-\alpha)$ are approximately dual objects, similar to the duality between $L(s,\chi)$ and $L(s,\overline{\chi})$.

[10] possibly because he wants to reserve the phrase "functional equation of the Hurwitz zeta function" for [6, Theorem 12.8]

1.2 Moments of the Riemann zeta function

To put our discussion of the moments of the Hurwitz zeta function in the context of the literature, it behooves us to discuss the rich history of moments (or mean-values) of $\zeta(s)$ and other L-functions from arithmetic.

The moments of the Riemann zeta function are defined by

$$M_k(T) = \int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} \, dt, \tag{1.6}$$

and are a classical topic of consideration – they were first considered by Hardy and Littlewood in the early 1900s. The reader is invited to peruse [81, Chapter V] for an account of the classical topics in this theory.

Perhaps the biggest reason for the interest in these problems is the connection to the Lindelöf hypothesis; one has that

$$M_k(T) \ll_{k,\epsilon} T^{1+\epsilon} \text{ for every } k \in \mathbb{N} \iff \zeta(\tfrac{1}{2}+it) \ll_\epsilon t^\epsilon,$$

where the upper bounds are respectively as $T, t \to \infty$. The statement $\zeta(\frac{1}{2}+it) \ll t^\epsilon$ is the well-known Lindelöf hypothesis, which follows from the Riemann hypothesis[11] but is not known unconditionally; the Lindelöf hypothesis has implications for zero-density estimates, which themselves have implications for gaps between primes – see, for example, [51, §§1.9, 12.5] for an involved discussion of these matters.

The following conjecture is widely believed:

Conjecture 1.2.1 (folklore)**.** *Suppose $k \geqslant 0$ is a fixed real number, and $T \to \infty$, then*

$$M_k(T) \sim c_k T (\log T)^{k^2},$$

for some fixed positive constant c_k depending only on k.

[11] henceforth abbreviated as RH

This conjecture is trivial for $k = 0$, was proved by Hardy and Littlewood [33] for $k = 1$, was proved by Ingham [48] for $k = 2$, and is wide open in all other cases.

Despite the history and intractability of the problem, very precise conjectures for the exact value of c_k exist. On the basis of number theoretic calculations, Conrey and Ghosh [21] conjectured the value of c_k for $k = 3$ and by a different (but still number theoretic) method, Conrey and Gonek [22] conjectured the value of c_k for $k = 3, 4$. Finally, using a heuristic modeling $\zeta(s)$ by characteristic polynomials of random matrices from the circular unitary ensemble, Keating and Snaith [55] conjectured the value of c_k for all $k > 0$, agreeing with the conjectures from [21] and [22].

The method of Conrey and Gonek mentioned above depends on conjectures about the additive divisor problem, namely, the question of getting asymptotics for

$$\sum_{n \leqslant X} d_k(n) d_k(n + r),$$

with some uniformity in r. Here, k is a fixed integer and $d_k(n)$ is the coefficient of $\zeta(s)^k$. The additive divisor problem is trivial when $k = 1$, and tractable[12] when $k = 2$ (see [64] and references therein). For $k \geqslant 3$, however, the problem is currently out of reach. This is one explanation for why we know the low moments, $k = 1, 2$, but do not know any higher moments with $k \geqslant 3$. The reader is referred to [65] (respectively [66]) for a recent attempt at conditionally formalizing Conrey–Gonek's heuristic argument for $k = 3$ (respectively $k = 4$).

The analogy with random matrix theory has led to many fruitful conjectures for moments of L-functions; see, for example, [20] and the references therein.

[12] using a connection to Kloosterman sums due to Estermann [27], whereby tools from algebraic geometry and the spectral theory of automorphic forms come into play – note however Ingham [49] had asymptotics with log-savings already in the 1920s

In §1.2.1, we review some of the connections between L-functions and random matrices.

A weaker, and hence theoretically more tractable version of Conjecture 1.2.1 is the estimate $M_k(T) \asymp_k T(\log T)^{k^2}$. By work of Ramachandra [70, 71, 72], and Heath-Brown [45], the lower bound $M_k(T) \gg_k T(\log T)^{k^2}$ was known conditionally on the Riemann Hypothesis (RH) for $k > 0$, and by work of Radziwiłł and Soundararajan [69], it was known unconditionally for all $k \geqslant 1$. Recent work of Heap and Soundararajan [43] establishes the lower bound unconditionally for all $k > 0$.

For the upper bound, Soundararajan [78] had shown on RH that $M_k(T) \ll_{k,\epsilon} T(\log T)^{k^2+\epsilon}$ for every $\epsilon > 0$ and $k > 0$. Harper [34] removed the dependence on ϵ, conditionally establishing the sharp upper bound for every $k > 0$. The upper bound was known unconditionally for $k = 1/n$, $n \in \mathbb{N}$ due to Heath-Brown [45], and for $k = 1 + 1/n$, $n \in \mathbb{N}$ due to Bettin, Chandee and Radziwiłł [8]. Recently, Heap, Radziwiłł and Soundararajan [40] subsumed both of these results by proving the upper bound unconditionally for $0 \leqslant k \leqslant 2$.

Many of the results discussed above generalize to other L-functions, both over number fields and over function fields. We refer the reader to Soundararajan's ICM 2022 plenary talk [79] for a survey of questions in the value distribution of L-functions, connections to random matrix theory, and recent progress (including some outlined above).

1.2.1 Connections to random matrix theory

The esoteric connection between the theory of L-functions and the theory of random matrices was discovered by coincidence in a tea-time discussion in the 70s between Hugh Montgomery and Freeman Dyson, when Dyson observed that Montgomery's conjectural description [63] of the pair statistics of the zeroes of $\zeta(s)$

agrees with the corresponding statistics for the eigenvalues of random unitary matrices.

While this connection is purely conjectural in the number field setting, there is a mountain of evidence supporting it, both theoretical and numerical. The numerical evidence is largely due to the work of Odlyzko [67], who computed many high zeroes of $\zeta(s)$ to test what is now called the GUE hypothesis. The theoretical evidence is even more convincing when one looks at L-functions beyond $\zeta(s)$ – for example:

- Hejhal [46] computed the triple correlation statistics for the zeroes of $\zeta(s)$ and showed they agreed with the GUE hypothesis. Rudnick and Sarnak [74] generalized this and showed that, actually, all n-level correlation statistics high up on the critical line for all principal automorphic L-functions universally agree with the GUE hypothesis, and are not sensitive to the coefficients of the L-functions.

- Perhaps most convincing is the evidence from the function field setting. Here, Katz and Sarnak [53] considered the statistics of low-lying zeroes in families of L-functions and – going well-beyond the Riemann Hypothesis of these L-functions[13] – rigorously proved that these statistics are modeled by those of random matrices from the classical compact groups[14]). On the basis of this, they formulated what is now called the Katz–Sarnak philosophy; namely, they conjectured that a similar phenomenon holds for families of L-functions over number fields.

- Support for the Katz–Sarnak philosophy was provided by seminal work of Iwaniec, Luo, and Sarnak [52] who computed many of these statistics in

[13] known due to work of Deligne

[14] The statistic for L-functions depends only on the compact group that arises; for this reason, the matrix group associated to a family is called its "symmetry type".

automorphic families over number fields both unconditionally and under several unproved hypotheses, showing that they agree with the philosophy.

We refrain from further explanations of the above results as that would lead us far afield from our goal. We remark, however, that for a fixed L-function, high up in the critical strip one universally expects GUE behaviour – the zeroes of the L-function are modeled by eigenvalues of random unitary matrices. This will be relevant in §2.5.

All the connections noted above are about zeroes of L-functions. Since the eigenvalues of a matrix are zeroes of its characteristic polynomial, one may wonder if the correspondence goes deeper: perhaps, L-functions correspond to characteristic polynomials of random matrices. This idea led Keating and Snaith to the following refinement of Conjecture 1.2.1.

Conjecture 1.2.2 (Keating–Snaith [55])**.** *For fixed $k \in \mathbb{C}$ with $\Re(k) > -1/2$,*

$$M_k(T) \sim a(k)f(k)T(\log T)^{k^2},$$

where

$$a(k) = \prod_p \left\{ \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^m} \right\}, \tag{1.7}$$

$$f(k) = \frac{G^2(k+1)}{G(2k+1)}. \tag{1.8}$$

Here and throughout, d_k is the k-fold divisor function[15] *and G is the Barnes G-function.*

To explain the key insight of Keating and Snaith, namely, that

$$c_k = a(k)f(k),$$

[15] i.e., the coefficient of n^{-s} in the Dirichlet series of $\zeta(s)^k$

with $a(k)$ as in (1.7) and $f(k)$ as in (1.8), it is illuminating to recall the backdrop in which they made this conjecture. Conrey and Ghosh [21] were the first to refine the folkloric Conjecture 1.2.1, by suggesting that when k is an integer, then

$$c_k = \frac{a(k)g(k)}{k^2!}.$$

where $a(k)$ is as in (1.7), which they called the "arithmetic factor" and $g(k)$ is as an integer, which they called the "geometric factor"[16]. It is not hard to guess that an Euler product like $a(k)$ must occur[17], so there was some mystery behind the provenance and nature of $g(k)$. In this framework, Hardy and Littlewood's result [33] proves $g(1) = 1$, while Ingham's result [48] proves $g(2) = 2$. Additionally, Conrey–Ghosh conjectured that $g(3) = 42$. Then, using the divisor correlation heuristic described earlier, Conrey–Gonek [22] conjectured that $g(4) = 24024$. Conrey–Gonek's work recovered previously known values for $k \in \{1, 2, 3\}$, but failed for $k > 4$ – in fact, it gave a negative value of $g(5)$!

Keating and Snaith's insight was the following. First, since the Gaussian unitary ensemble (GUE) and the circular unitary ensemble (CUE) have identical eigenvalue distributions in the large dimension limit, one works with CUE instead of GUE. If U is an $N \times N$ unitary matrix with eigenvalues $\{e^{i\theta_n}\}_{n=1}^N$, the characteristic polynomial, $Z(U, \theta)$ is defined by

$$Z(U, \theta) = \prod_{n=1}^{N} (1 - e^{i(\theta_n - \theta)}).$$

Then, they proved that

$$\mathbb{E}[|Z(U, \theta)|^{2k}] \sim f(k) N^{k^2},$$

[16] In the recent literature, it is common to call this the "random matrix" factor instead. However, as we shall presently explain, one should probably call a_k the "prime contribution" and g_k the "zeroes contribution".

[17] It arises from a pole when using Perron's formula to calculate the contributions of the diagonal terms after an application of the approximate functional equation.

when $N \to \infty$, where here $f(k)$ is as in (1.8), and the expectation is taken by sampling U from the unitary group $\mathfrak{U}(N)$ uniformly with respect to the Haar measure. Now, observe that eigenvalues of U have mean spacing $2\pi/N$ while zeroes of $\zeta(s)$ have mean spacing $2\pi/\log T$. Thus, if we take $N = \log T$, one might expect $Z(U,\theta)$ to be a good model for $\zeta(s)$, and hence that

$$\mathbb{E}[|Z(U,\theta)|^{2k}] \simeq \frac{M_k(T)}{T},$$

in some sense. They observed that

$$f(k) = \frac{g(k)}{k^2!}$$

for the known values ($k = 1,2$) and the conjectured values ($k = 3,4$) of $g(k)$ and hence were led to guess that that the above[18] continues for all $k \in \mathbb{C}$ with $\Re(k) > -1/2$, which is their conjecture above.

This spectacular piece of guesswork is remarkably effective – Keating and Snaith were able to extend this line of thought also to moments in families of L-functions [54].

A skeptical reader may, however, ask: where is the constant a_k coming from? Its inclusion in Conjecture 1.2.2 appears ad-hoc; it does not seem to be the case that primes are involved at all. Since $\zeta(s)$ is, at its core, an arithmetic object, it is unclear how to reconcile the arithmetic of $\zeta(s)$ with the random matrix theory analogy.

This discrepancy was explained by the hybrid formula of Gonek, Hughes, and Keating [31]. Recall that,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \frac{e^{\alpha+\beta s}}{s(s-1)\Gamma(\frac{s}{2})} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where α and β are constants, p runs over all primes, and ρ runs over all non-trivial zeroes of $\zeta(s)$. The former representation is the Euler product, while the latter

[18] with $k^2!$ replaced by $\Gamma(k^2+1)$

is the Hadamard product[19] of $\zeta(s)$. Gonek, Hughes, and Keating formalulated a hybrid Euler-Hadamard product involving both the primes and the zeroes. This formula is pretty technical, so with the aim of simplifying the exposition, we state and use it here informally; the reader interested in the gory details may refer to Chapter 2, or to the original article [31]. With this in mind, the formula states

$$\zeta(s) \simeq P_X(s) Z_X(s)$$

where X is a parameter that grows as $t = \Im(s)$ grows in a vertical strip, but which can be chosen with a fair amount of flexibility. Here, P_X is essentially a partial Euler product using only primes up to X, while Z_X is essentially a partial Hadamard product using zeroes near t, viz., $|\rho - t| \ll \frac{1}{\log X}$. Note that there is an uncertainty principle at play here – to capture the true behaviour of $\zeta(s)$, one has to pay the price somewhere. If we choose to use fewer primes in the Euler product, we need more zeroes in the Hadamard product, and conversely.

With this formula in hand, Gonek, Hughes, and Keating conjectured[20], that as $X, T \to \infty$ with X growing sufficient slowly with respect to T, $P_X(s)$ and $Z_X(s)$ behave like independent random variables, and hence the moments factorize,

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \sim \frac{1}{T}\int_T^{2T} |P_X(\tfrac{1}{2}+it)|^{2k}\,dt \times \frac{1}{T}\int_T^{2T} |Z_X(\tfrac{1}{2}+it)|^{2k}\,dt.$$

This is called the Gonek–Hughes–Keating splitting conjecture.

The moments of P_X can be computed exactly using standard arguments; it turns out to be

$$\frac{1}{T}\int_T^{2T} |P_X(\tfrac{1}{2}+it)|^{2k}\,dt \sim a(k)(e^{\gamma}\log X)^{k^2}.$$

On the other hand, we do not know how to compute the moments of Z_X. However, following Keating and Snaith, one can model Z_X by a corresponding object involving eigenvalues of a random unitary matrix with dimension $N = \log T$. Upon

[19] Strictly speaking, it is the Hadamard product of the completed zeta function, $\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, which is entire of order 1.

[20] on the basis of a separation of scales

doing so, and computing the corresponding moment in the random matrix theory setting, Gonek, Hughes, and Keating conjectured that

$$\frac{1}{T}\int_T^{2T} |Z_X(\tfrac{1}{2}+it)|^{2k}\,dt \sim f(k)\left(\frac{\log T}{e^\gamma \log X}\right)^{k^2}.$$

But now, the previous three display equations combine to give

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \sim a(k)f(k)(\log T)^{k^2},$$

recovering the Keating and Snaith conjecture! It seems surprising that the scale parameter X cancels exactly in the formula. This approach naturally includes the contributions from primes[21], providing further evidence of the correctness of the Keating–Snaith conjectures, and it is the approach we shall follow in Chapter 2 when we consider moments of products of Dirichlet L-functions.

[21] and, as mentioned earlier, justifies our suggestion to call g_k the "zeroes contribution"

2 Moments of products of Dirichlet L-functions

2.1 Introduction

Due to (1.1), moments of products of Dirichlet L-functions (to the same modulus, q) arise naturally when one seeks to study $M_k(T;\alpha)$ with $\alpha \in \mathbb{Q}$. To explain this connection, we fix some notation that will be used throughout this chapter and its sequel. We assume $\alpha = a/q$ with[1] $q \geqslant 3$, $(a,q) = 1$, and $1 \leqslant a \leqslant q$. Dirichlet characters will be denoted χ or ν, and will be modulo q unless noted otherwise. We will use bolded, lower case (Greek or Latin) letters such as $\boldsymbol{\ell}$ for tuples of natural numbers indexed by characters modulo q. Thus, if $\boldsymbol{\ell}$ is such a tuple, we think of it as a function $\boldsymbol{\ell} : \mathcal{D}(q) \to \mathbb{N}$ where $\mathcal{D}(q)$ is the set of Dirichlet characters modulo q. We denote $\boldsymbol{\ell}(\chi)$ as ℓ_χ. Further, we define,

$$|\boldsymbol{\ell}| = \sum_{\chi} \ell_\chi, \qquad \lambda(\boldsymbol{\ell}) = \sum_{\chi} \ell_\chi^2, \qquad \mathcal{L}^{\boldsymbol{\ell}}(s) = \prod_{\chi} L(s,\chi)^{\ell_\chi}.$$

Here, and later, sums and products over χ or ν run over $\mathcal{D}(q)$. If $\boldsymbol{\ell}$ is clear from context, we suppress it and denote $\lambda(\boldsymbol{\ell})$ simply as λ. Finally, we denote by $d_{\boldsymbol{\ell}}(n)$ the coefficient of n^{-s} in the Dirichlet series expansion of $\mathcal{L}^{\boldsymbol{\ell}}(s)$.

[1] We may assume $q \geqslant 3$, since $\zeta(s;1) = \zeta(s)$ and $\zeta(s,\frac{1}{2}) = (2^s - 1)\zeta(s)$.

Now, by raising (1.1) to the power $2k$, and using the multinomial theorem

$$\begin{aligned}|\zeta(s,\alpha)|^{2k} &= \left| \frac{q^{ks}}{\varphi(q)^k} \sum_{|\boldsymbol{\ell}|=k} \binom{k}{\boldsymbol{\ell}} \prod_{\chi} \left\{ \overline{\chi}(a) L(s,\chi) \right\}^{\ell_\chi} \right|^2 \\ &= \left| \frac{q^{ks}}{\varphi(q)^k} \sum_{\substack{|\boldsymbol{\ell}^{(1)}|=k \\ |\boldsymbol{\ell}^{(2)}|=k}} \binom{k}{\boldsymbol{\ell}} \Big\{ \prod_{\chi} \overline{\chi}(a)^{\ell_\chi} \Big\} \boldsymbol{\ell}) \mathcal{L}^{\boldsymbol{\ell}}(s) \right|^2 \\ &= \frac{q^{2k\sigma}}{\varphi(q)^{2k}} \sum_{\substack{|\boldsymbol{\ell}^{(1)}|=k \\ |\boldsymbol{\ell}^{(2)}|=k}} \binom{k}{\boldsymbol{\ell}^{(1)}} \binom{k}{\boldsymbol{\ell}^{(2)}} s(a;\boldsymbol{\ell}^{(1)},\boldsymbol{\ell}^{(2)}) \mathcal{L}^{\boldsymbol{\ell}^{(1)}}(s) \overline{\mathcal{L}^{\boldsymbol{\ell}^{(2)}}(s)},\end{aligned} \tag{2.1}$$

where $\binom{k}{\boldsymbol{\ell}} = k!/\prod_\chi \ell_\chi!$ are multinomial coefficients, the sums runs over $\boldsymbol{\ell}$ such that $|\boldsymbol{\ell}| = \sum_\chi \ell_\chi = k$, and $s(a;\boldsymbol{\ell}^{(1)},\boldsymbol{\ell}^{(2)}) = \prod_\chi \chi(a)^{\ell_\chi^{(2)} - \ell_\chi^{(1)}}$. In particular, when we integrate both sides from $1/2 + iT$ to $1/2 + i2T$, the terms in this sum whose phase oscillates will probably not contribute to the main term. The terms that do not have oscillations correspond to the diagonal terms $\boldsymbol{\ell}^{(1)} = \boldsymbol{\ell}^{(2)}$ where the phases of each term in the product cancel out, yielding a positive real number. Thus, heuristically,

$$\begin{aligned}M_k(T;\alpha) &= \int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it,\alpha\right)\right|^{2k} dt \\ &\approx \frac{q^k}{\varphi(q)^{2k}} \sum_{|\boldsymbol{\ell}|=k} \binom{k}{\boldsymbol{\ell}}^2 \int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}\left(\tfrac{1}{2}+it\right)\right|^2 dt.\end{aligned} \tag{2.2}$$

whence, the problem of estimating $M_k(T;\alpha)$ naturally reduces to studying the mean-square of $\mathcal{L}^{\boldsymbol{\ell}}(s)$ along the critical line.

The mean-square of $\mathcal{L}^{\boldsymbol{\ell}}(s)$ has been considered in the literature before. The principal reason for interest in such products is the connection to Dedekind zeta functions $\zeta_K(s)$ for Abelian number fields K. For example, if

$$\boldsymbol{\ell} = k\mathbb{1},$$

or, in other words, $\ell_\chi = k$ for every χ, then

$$\mathcal{L}^{\boldsymbol{\ell}}(s) = \zeta_K(s)^k,$$

where $K = \mathbb{Q}(\epsilon_q)$ is the qth cylotomic field (i.e., ϵ_q is a primitive qth root of unity), and, hence, we find that the mean-square of $\mathcal{L}^{\boldsymbol{\ell}}(s)$ is the $2k$th moment of $\zeta_K(s)$. On the other hand, if $K = \mathbb{Q}(\sqrt{d})$, D is the discriminant of this field, and $\chi_d(n) = (D|n)$ where $(\cdot|\cdot)$ is the Kronecker symbol, then one finds that if we set

$$\ell_\chi = \begin{cases} k & \text{if } \chi = \chi_0, \chi_d \\ 0 & \text{otherwise,} \end{cases}$$

then, $\mathcal{L}^{\boldsymbol{\ell}}(s)$ is essentially $\zeta_K(s)^k$ (after correcting the local factors at primes p dividing D). This is because,

$$\zeta_K(s) = \zeta(s) L(s, \chi_d),$$

which is the same as $L(s, \chi_0) L(s, \chi_d)$ up to the local factors corresponding to p dividing D.

We would like to highlight the following previous works on moments of products of L-functions on the critical line:

- Heap [38]: in which he studies the Dedekind zeta function $\zeta_K(s)$ of Galois number fields using the Gonek–Hughes–Keating approach, and modifies the recipe of [20] to give a conjecture for moments of products of L-functions in the Selberg class satisfying Selberg's orthonormality conjecture.

- Milinovich and Turnage-Butterbaugh [61]: in which they prove almost[2] sharp upper bounds for moments of products of automorphic L-functions under the generalized Riemann hypothesis (GRH) for the relevant L-functions by adapting the argument of [78].

- Topacogullari [82]: in which he proves asymptotic formulas with the best-known error terms for the mean-square of $L(s, \chi) L(s, \nu)$ where χ and ν are primitive characters to possibly different moduli, including the case where $\nu = \chi$.

[2] up to a factor of $(\log T)^\epsilon$, which can likely be removed by using ideas of Harper [34]

The connection to our work is hopefully evident – for example, Dirichlet L-functions are both automorphic L-functions of degree 1 and examples of the Selberg class known to satisfy the orthonormality conjecture. The reader is referred to the introduction to these articles for a more detailed account of the literature on moments of products. In §2.2, we collect the statements, as lemmata, of some of these results that we will need.

To study moments of products of Dirichlet L-functions, we will use a hybrid Euler-Hadamard product, a tool introduced originally by Gonek, Hughes and Keating [31] in the context of the Riemann zeta function, and discussed briefly in §1.2.1. Specifically, we will need the following version for Dirichlet L-functions in the t-aspect:

Theorem 2.1.1. *Let $s=\sigma+it$ with $\sigma \geqslant 0$ and $|t| \geqslant 2$, let $X \geqslant 2$ be a real parameter, and let K be any fixed positive integer. Further, let $f(x)$ be a non-negative C^∞-function of mass one supported on $[0,1]$, and set $u(x) = Xf(X\log(x/e)+1)/x$ so that u is a non-negative C^∞-function of mass one supported on $[e^{1-1/X}, e]$. Set*

$$U(z) = \int_0^\infty u(x) E_1(z\log x)\,dx,$$

where $E_1(z) = \int_z^\infty e^{-w}w^{-1}\,dw$ is the exponential integral.

Let q be a fixed positive integer, and χ be a Dirichlet character modulo q with conductor $q^*(\chi)$. Further, suppose that χ is induced by the primitive character χ^* modulo $q^*(\chi)$. Then,

$$L(s,\chi) = P_X(s,\chi)Z_X(s,\chi)\left(1 + O\Big(\frac{\log X}{X^\sigma}\Big) + O_{K,f}\Big(\frac{X^{K+2}}{(|s|\log X)^K}\Big)\right),$$

where

$$P_X(s,\chi) = \left\{\prod_{p|q}\left(1-\frac{\chi^*(p)}{p^s}\right)\right\}\exp\left(\sum_{n\leqslant X}\frac{\chi^*(n)\Lambda(n)}{n^s\log n}\right),$$

and

$$Z_X(s,\chi) = \exp\left(-\sum_{\substack{\rho \\ 0\leqslant\Re\rho\leqslant 1 \\ L(\rho,\chi^*)=0}} U((s_0-\rho)\log X)\right).$$

The implied constants are uniform in all parameters including q, unless indicated otherwise.

Such a hybrid Euler-Hadamard product was proved by Bui and Keating [16] in their study of moments in the q-aspect of Dirichlet L-functions at the central point $s = 1/2$ (see [16, Remark 1]). Similar hybrid Euler-Hadamard products have been used in the literature for studying moments in many other contexts such as for for orthogonal and symplectic families of L-functions [17]; for $\zeta'(s)$ [15]; for the Dedekind zeta function $\zeta_K(s)$ of a Galois extension K of $\mathbb{Q}$ [38]; for quadratic Dirichlet L-functions over function fields [14], [4]; for normalized symmetric square L-functions associated with $SL_2(\mathbb{Z})$ eigenforms [25]; and for quadratic Dirichlet L-functions over function fields associated to irreducible polynomials [3].

With $P(s,\chi)$ and $Z(s,\chi)$ as in Theorem 2.1.1, we define

$$\mathcal{P}_X^{\boldsymbol{\ell}}(s) = \prod_{\chi} P_X(s,\chi)^{\ell_\chi},\ \mathcal{Z}_X^{\boldsymbol{\ell}}(s) = \prod_{\chi} Z_X(s,\chi)^{\ell_\chi}.$$

We can view $\mathcal{L}^{\boldsymbol{\ell}}(s)$ as an L-function of degree $|\boldsymbol{\ell}|$, $\mathcal{P}_X^{\boldsymbol{\ell}}(s)$ as an approximation to its Euler product, and $\mathcal{Z}_X^{\boldsymbol{\ell}}(s)$ as an approximation to its Hadamard product. Roughly, Theorem 2.1.1 implies that $\mathcal{L}^{\boldsymbol{\ell}}(s) \approx \mathcal{P}_X^{\boldsymbol{\ell}}(s)\mathcal{Z}_X^{\boldsymbol{\ell}}(s)$.

As is usually the case with hybrid Euler-Hadamard products, X mediates between the primes and zeroes; if we want to take fewer primes in the Euler product we must take more zeroes in the Hadamard product and vice-versa.

For X growing relatively slowly with T, we expect the two terms in the decomposition $\mathcal{L}^{\boldsymbol{\ell}}(s) \approx \mathcal{P}_X^{\boldsymbol{\ell}}(s)\mathcal{Z}_X^{\boldsymbol{\ell}}(s)$ to behave like independent random variables due to a separation of scales. This is analogous to the splitting conjecture of Gonek, Hughes and Keating [31, Conjecture 2]. Concretely, we have:

Conjecture 2.1.2 (Splitting)**.** *Let $X, T \to \infty$ with $X \ll_\epsilon (\log T)^{2-\epsilon}$. Then, for any tuple of nonnegative integers $\boldsymbol{\ell}$ indexed by characters modulo q, we have for*

$s = 1/2 + it$,

$$\frac{1}{T}\int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}(s)\right|^2 dt \sim \left(\frac{1}{T}\int_T^{2T} \left|\mathcal{P}_X^{\boldsymbol{\ell}}(s)\right|^2 dt\right) \times \left(\frac{1}{T}\int_T^{2T} \left|\mathcal{Z}_X^{\boldsymbol{\ell}}(s)\right|^2 dt\right).$$

On [31, p. 511], it is suggested that this splitting conjecture holds for a much wider range of X and T with $X = o(T)$. Recently, Heap [39] has justified this suggestion. He proved on RH that the splitting conjecture for $\zeta(s)$ holds for every $k > 0$ and a much wider range of X provided one requires only an order of magnitude result, instead of an asymptotic. He also established the splitting conjecture for $k = 1$ and $k = 2$ for wider ranges of X both with and without RH.

The mean-square of $\mathcal{P}_X^{\boldsymbol{\ell}}(s)$ can be computed exactly.

Theorem 2.1.3. *Let $k \geqslant 0$ be a fixed integer and $\epsilon > 0$ be fixed. Let $\boldsymbol{\ell}$ be a tuple of nonnegative integers indexed by characters modulo q such that $|\boldsymbol{\ell}| = \sum_\chi \ell_\chi = k$. Finally, suppose that $q^2 < X \ll_\epsilon (\log T)^{2-\epsilon}$. Then for $s = 1/2 + it$,*

$$\frac{1}{T}\int_T^{2T} |\mathcal{P}_X^{\boldsymbol{\ell}}(s)|^2 \, dt = b(\boldsymbol{\ell}) F_X(\boldsymbol{\ell}) \left(1 + O_{q,k,\epsilon}\left(\frac{1}{\log X}\right)\right)$$

where $b(\boldsymbol{\ell})$ and $F_X(\boldsymbol{\ell})$ are given by

$$b(\boldsymbol{\ell}) = \prod_{p \nmid q} \left\{ \left(1 - \frac{1}{p}\right)^{|d_{\boldsymbol{\ell}}(p)|^2} \sum_{m=0}^{\infty} \frac{|d_{\boldsymbol{\ell}}(p^m)|^2}{p^m} \right\}, \tag{2.3}$$

$$F_X(\boldsymbol{\ell}) = (e^\gamma \log X)^\lambda \prod_p \left(1 - \frac{1}{p}\right)^{\lambda - |d_{\boldsymbol{\ell}}(p)|^2}, \tag{2.4}$$

where γ is the Euler-Mascheroni constant, $d_{\boldsymbol{\ell}}(n)$ is the coefficient of n^{-s} in the Dirichlet series for $\mathcal{L}^{\boldsymbol{\ell}}(s)$, and $\lambda = \sum_\chi \ell_\chi^2$.

One could prove a similar result uniformly in c on any vertical line $\Re s = \sigma$ with $1 > \sigma \geqslant c \geqslant 1/2$ given $X \ll_\epsilon (\log T)^{1/(1-c+\epsilon)}$, but we choose not to do so for conciseness. Note that the product over p in (2.4) is conditionally convergent but not absolutely convergent.

For the mean-square of $\mathcal{Z}_X^{\boldsymbol{\ell}}(s)$, we use random matrix theory to model each L-function appearing in the product by random unitary matrices. One expects that the matrices representing distinct L-functions behave independently as in [38, Conjecture 2]. This leads to:

Conjecture 2.1.4. *Suppose that $X, T \to \infty$ with $X \ll_\epsilon (\log T)^{2-\epsilon}$. Then, for any tuple $\boldsymbol{\ell}$ of nonnegative integers indexed by characters modulo q, we have for $s = 1/2 + it$,*

$$\frac{1}{T}\int_T^{2T} |\mathcal{Z}_X^{\boldsymbol{\ell}}(s)|^2\,dt \sim \prod_\chi \left[\frac{G(\ell_\chi+1)^2}{G(2\ell_\chi+1)}\left(\frac{\log q^*(\chi)T}{e^\gamma \log X}\right)^{\ell_\chi^2}\right],$$

where $G(\cdot)$ is the Barnes G-function, and $q^*(\chi)$ is the conductor of χ.

It is clear that one can use Conjectures 2.1.2 and 2.1.4 together with Theorem 2.1.3 to get a conjectural asymptotic for $\int_T^{2T} |\mathcal{L}^{\boldsymbol{\ell}}(1/2+it)|^2\,dt$. Precisely, we get,

Theorem 2.1.5. *If Conjecture 2.1.2 and Conjecture 2.1.4 are true for a tuple of nonnegative integers $\boldsymbol{\ell}$ indexed by characters modulo q satisfying $|\boldsymbol{\ell}| = k$, then we have for $s = 1/2 + it$,*

$$\frac{1}{T}\int_T^{2T} |\mathcal{L}^{\boldsymbol{\ell}}(s)|^2\,dt = (c_{\boldsymbol{\ell}}(q) + o_{q,k}(1))\left\{\prod_\chi (\log q^*(\chi)T)^{\ell_\chi^2}\right\},$$

where $c_{\boldsymbol{\ell}}(q)$ is given by

$$\prod_p \left\{\left(1-\frac{1}{p}\right)^\lambda \sum_{m=0}^\infty \frac{|d_{\boldsymbol{\ell}}(p^m)|^2}{p^m}\right\}\prod_\chi \frac{G(\ell_\chi+1)^2}{G(2\ell_\chi+1)}.$$

Here λ, and $G(\cdot)$ and $q^*(\chi)$ are the same as above.

Note that for a fixed q, the above says that the mean-square of a product of Dirichlet L-functions grows as $\asymp_{k,q} T(\log T)^\lambda$. This is known for $|\boldsymbol{\ell}| \leqslant 2$, and we shall show that in these cases our predicted constant matches up.

Due to the conditional hypotheses, the above theorem is really a conjecture. We note here that Heap made a similar conjecture about moments of products of L-functions from the Selberg class (see [38, §6]) using the recipe of Conrey, Farmer, Keating, Rubinstein and Snaith [20]. Specializing to Dirichlet L-functions, one can recover the above conjecture.

He also discussed how such conjectures could be reproduced by using hybrid Euler-Hadamard products under appropriate hypotheses. However, since he has not worked out the details of this approach in this specific context, we do so here for completeness.

Since the current levels of technology can handle second moments and fourth moments of $\zeta(s)$ really well, it is natural to hope that we can prove Conjectures 2.1.2 and 2.1.4 for $|\boldsymbol{\ell}| \leqslant 2$. We define the Kronecker delta $\boldsymbol{\delta}^{\chi}$ by

$$\delta^{\chi}_{\nu} = \begin{cases} 1 & \text{if } \chi = \nu \\ 0 & \text{if } \chi \neq \nu. \end{cases}$$

Then, we can prove:

Theorem 2.1.6. *Conjecture 2.1.2 and Conjecture 2.1.4 hold unconditionally for* $|\boldsymbol{\ell}| = 1$. *In particular* $|\boldsymbol{\ell}| = 1$ *if and only if* $\boldsymbol{\ell} = \boldsymbol{\delta}^{\chi}$ *for some character* χ, *in which case we have that for* $s = 1/2 + it$, *and* $X, T \to \infty$ *with* $X \ll_{\epsilon} (\log T)^{2-\epsilon}$,

$$\frac{1}{T}\int_T^{2T} |L(s,\chi)|^2 \, dt \sim \left(\frac{1}{T}\int_T^{2T} |P_X(s,\chi)|^2 \, dt\right) \times \left(\frac{1}{T}\int_T^{2T} |Z_X(s,\chi)|^2 \, dt\right),$$

and

$$\frac{1}{T}\int_T^{2T} |Z_X(s,\chi)|^2 \, dt \sim \frac{\log q^*(\chi) T}{e^{\gamma}\log X}. \tag{2.5}$$

The above theorem can almost certainly be extended to the case $|\boldsymbol{\ell}| = 2$. This corresponds to $\boldsymbol{\ell} = \boldsymbol{\delta}^{\chi} + \boldsymbol{\delta}^{\nu}$, and $\mathcal{L}^{\boldsymbol{\ell}}(s) = L(s,\chi)L(s,\nu)$ with χ and ν not necessarily distinct characters modulo q.

We note first that some of these have already been proved. The case $\boldsymbol{\ell} = 2\boldsymbol{\delta}^{\chi_0}$ where χ_0 is the principal character modulo q was essentially proved by Gonek, Hughes, and Keating [31, Theorem 3]. More generally, the case $\boldsymbol{\ell} = \boldsymbol{\delta}^{\chi_0} + \boldsymbol{\delta}^{\chi}$ where χ is a (not necessarily primitive) quadratic Dirichlet character modulo q was essentially proved by Heap [38, Theorem 3]. To see this, note from (2.9) that $Z_X(s, \chi)$ depends only on the primitive character χ^* modulo $q^*(\chi)$ that induces χ. In particular, one can replace $L(s, \chi_0)^2$ with $\zeta(s)^2$ and $L(s, \chi_0)L(s, \chi)$ with $\zeta(s)L(s, \chi^*) = \zeta_K(s)$ where K is a quadratic extension of $\mathbb{Q}$ and $\zeta_K(s)$ is its Dedekind zeta function. Analogues of splitting for these products is precisely what was proven in these papers.

By following both these arguments, one should be able to extend to the general case $\boldsymbol{\ell} = \boldsymbol{\delta}^{\chi} + \boldsymbol{\delta}^{\nu}$. To do so, one would need a moment result for the product of two primitive Dirichlet L-functions and a short Dirichlet polynomial, generalizing that of [36]. That is, we would need an asymptotic for

$$\int_T^{2T} \left| L(s, \chi)L(s, \nu) \sum_{n \leqslant T^{\theta}} \frac{a_n}{n^s} \right|^2 dt, \tag{2.6}$$

where χ and ν are any primitive characters with conductor dividing q, and some $0 < \theta < 1$ sufficiently large. Such asymptotics exist in the special cases of $\zeta(s)^2$ [47, 7] and $\zeta(s)L(s, \chi)$ [36], for any character χ. Proving (2.6) and the splitting conjecture for $\boldsymbol{\ell} = \boldsymbol{\delta}^{\chi} + \boldsymbol{\delta}^{\nu}$ for more general χ, ν by using the methods of [38], [36] and [31] as outlined above should be possible but long and technical. Thus, we do not pursue this here.

In several results, we must assume one of the following two hypotheses:

- The Generalized Riemann Hypothesis holds for $L(s, \chi)$ for every character χ modulo q. We denote this by $\mathsf{GRH}(q)$.

- The mean squares of $\mathcal{L}^{\boldsymbol{\ell}}(s)$ on the critical line for all $|\boldsymbol{\ell}| = k$ satisfy Conjecture 2.1.2 and Conjecture 2.1.4. We denote this by $\mathsf{Sp}(q, k)$, for splitting conjecture.

We introduce the above shorthand for convenience, as many results will hold under either hypothesis.

2.2 Previous results

In this section, we collect some previous results from the literature as lemmata.

We will need Mertens' theorem for arithmetic progressions:

Lemma 2.2.1. *Let κ be a fixed real number, and $(c,q)=1$. Then,*

$$\prod_{\substack{p\leqslant X \\ p\equiv c \pmod q}} \left(1-\frac{1}{p}\right)^{-\kappa} = H_c^q(\kappa)\left(1+O_{q,\kappa}\left(\frac{1}{\log X}\right)\right)$$

where,

$$H_c^q(\kappa) = \left\{ e^{\gamma}\log X \prod_p \left(1-\frac{1}{p}\right)^{1-\delta_q(p,c)\varphi(q)} \right\}^{\frac{\kappa}{\varphi(q)}}.$$

Here γ is the Euler-Mascheroni constant and $\delta_q(x,y)$ is the Kronecker delta in $\mathbb{Z}/q\mathbb{Z}$,

$$\delta_q(x,y) = \begin{cases} 1 & \textit{if } x\equiv y \pmod q, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Clearly the result for general $\kappa\in\mathbb{R}$ follows from the case $\kappa=1$ by exponentiating. The latter is precisely Merten's theorem for arithmetic progressions which was proved by Williams [85]. The expression for the constant $H_c^q(1)$, however, is due to Languasco and Zaccagnini [58, §6] who also improved the error term to one uniform in q. The weaker form suffices for our purposes.

$\square$

The following lemma is a corollary of a result of Milinovich and Turnage-Butterbaugh [61]:

Lemma 2.2.2. *Suppose that either* $\mathsf{GRH}(q)$ *or* $\mathsf{Sp}(q,k)$ *holds, and that* $\boldsymbol{\ell}$ *is tuple of nonnegative integers indexed by the characters modulo* q *satisfying* $|\boldsymbol{\ell}| = k$*. Then, for* $\lambda(\boldsymbol{\ell}) = \sum_\chi \ell_\chi^2$ *and any* $\epsilon > 0$*,*

$$\int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}\left(\tfrac{1}{2}+it\right)\right|^2 dt \ll_{q,k,\epsilon} T(\log T)^{\lambda+\epsilon}.$$

In particular, if $\boldsymbol{\ell} \neq k\boldsymbol{\delta}^\chi$ *for all characters* χ *modulo* q*, then*

$$\int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}\left(\tfrac{1}{2}+it\right)\right|^2 dt \ll_{q,k,\epsilon} T(\log T)^{k^2-1+\epsilon},$$

Proof. First, suppose that $\mathsf{Sp}(q,k)$ holds. Then, the first inequality is trivally true due to Theorem 2.1.5[3]. Alternatively, suppose that $\mathsf{GRH}(q)$ holds. Then, the first inequality follows by applying [61, Theorem 1.1] in the specific case where all the L-functions involved are Dirichlet L-functions.

Now note that under the constraints $\ell_\chi \geqslant 0$ and $\sum_\chi \ell_\chi = k$, we have that $\lambda = \sum_\chi \ell_\chi^2 \leqslant k^2$ with equality if and only if the entire weight of $\boldsymbol{\ell}$ is concentrated on a single character. In particular, if $\boldsymbol{\ell} \neq k\boldsymbol{\delta}^\chi$ for all characters χ, then $\lambda(\boldsymbol{\ell}) < k^2$ and so, $\lambda(\boldsymbol{\ell}) \leqslant k^2 - 1$. Thus, the second inequality in the lemma follows from the first.

□

We make use of a result of Topacogullari [82], where he computes the full asymptotic formula for the fourth moments of $L(s,\chi)$ and the mean-square of $L(s,\chi)L(s,\nu)$ with a power saving in the error term, and an explicit dependence on the conductors. We need only need a weak version of his results, stated below.

Lemma 2.2.3. *Let* χ *be a Dirichlet character modulo* q*. Then, for* $s = 1/2 + it$*,*

$$\int_T^{2T} |L(s,\chi)|^4 \, dt = C(\chi)T(\log T)^4 + O_q(T(\log T)^3)$$

[3] This is not circular; Lemma 2.2.2 is not used in the proof of Theorem 2.1.5.

where $C(\chi)$ *is given by*

$$C(\chi) = \frac{1}{2\pi^2}\frac{\varphi(q)^2}{q^2}\prod_{p|q}\left(1 - \frac{2}{p+1}\right).$$

Proof. This is an immediate corollary of [82, Theorem 1.1]. □

Lemma 2.2.4. *Let* χ *and* ν *be distinct Dirichlet characters modulo* q*. Then, for* $s = 1/2 + it$*,*

$$\int_T^{2T} |L(s,\chi)\,L(s,\nu)|^2 \; dt = D(\chi,\nu)T(\log T)^2 + O_q(T\log T)$$

where $D(\chi,\nu)$ *is given by*

$$D(\chi,\nu) = \frac{6}{\pi^2}|L(1,\chi\overline{\nu})|^2\frac{\varphi(q)}{q}\prod_{p|q}\left(1 - \frac{1}{p+1}\right).$$

Proof. This is a corollary of [82, Theorem 1.3], by setting $\chi_1 = \chi$, $\chi_2 = \nu$, $q_1 = q_2 = q$, noting that this implies $q_1^\star = q_2^\star = 1$ and noting that $\varphi(q^2) = q\varphi(q)$. □

We need a second moment asymptotic for a Dirichlet L-function twisted by a short Dirichlet polynomial. We use one proved by Wu [88]:

Lemma 2.2.5. *Let* χ *be a primitive Dirichlet character modulo* q *with* $\log q = o(\log T)$*, let* $\theta > 0$ *be a parameter, and let* $b(n)$ *be an arithmetic function satisfying* $b(n) \ll_\epsilon n^\epsilon$ *for all* $\epsilon > 0$*. Further, let*

$$B_\theta(s,\chi) = \sum_{n\leqslant T^\theta}\frac{\chi(n)b(n)}{n^s},$$

$$M_\theta(T;\chi,b) = \frac{1}{T}\int_T^{2T}\left|L\left(\tfrac{1}{2}+it,\chi\right)B_\theta\left(\tfrac{1}{2}+it,\chi\right)\right|^2 dt,$$

and

$$M'_\theta(T;\chi,b) = \frac{\varphi(q)}{q}\sum_{\substack{m,n\leqslant T^\theta\\(mn,q)=1}}\frac{b(m)\overline{b(n)}}{[m,n]}\left(\log\frac{qT(m,n)^2}{2\pi mn} + C + \sum_{p|q}\frac{\log p}{p-1}\right),$$

with $C = 2\gamma - 1 + 2\log 2$*. Then,*

$$M_\theta(T;\chi,b) = M'_\theta(T;\chi,b) + O(T^{-\varepsilon_\theta})$$

where the parameter ε_θ *depends on* θ*, and* $\varepsilon_\theta > 0$ *when* $\theta < 17/33$*.*

Proof. This is contained in [88, Theorem 1.1]. □

We also have the following, which is immediate from [16, Lemma 3]:

Lemma 2.2.6. *Let* $\boldsymbol{\ell}$ *be a tuple of nonnegative integers indexed by characters modulo* q *such that* $|\boldsymbol{\ell}| = \sum_\chi \ell_\chi = k$*, let*

$$P_X^*(s,\chi) = \prod_{p\leqslant X}\left(1 - \frac{\chi(p)}{p^s}\right)^{-1}\prod_{\sqrt{X}<p\leqslant X}\left(1 + \frac{\chi(p)^2}{2p^{2s}}\right)^{-1},$$

and let

$$\mathcal{P}_X^{*\boldsymbol{\ell}}(s) = \prod_\chi P_X^*(s)^{\ell_\chi}.$$

Then, uniformly for $\sigma \geqslant 1/2$ *and* $X > q^2$*,*

$$\mathcal{P}_X^{\boldsymbol{\ell}}(s) = \mathcal{P}_X^{*\boldsymbol{\ell}}(s)\left(1 + O_k\left(\frac{1}{\log X}\right)\right).$$

Proof. From [16, Lemma 3], we get that

$$P_X(s,\chi^*)^{\ell_\chi} = P_X^*(s,\chi^*)^{\ell_\chi}\left(1 + O_{\ell_\chi}\left(\frac{1}{\log X}\right)\right),$$

where χ^* is the primitive character modulo $q^*(\chi)$ which induces χ. Since $X > q^2$, we see that $p \mid q$ implies that $p \leqslant \sqrt{X}$. Thus, by inspection,

$$P_X^*(s,\chi) = P_X^*(s,\chi^*)\prod_{p|q}\left(1 - \frac{\chi^*(p)}{p^s}\right).$$

Putting the above two equalities together with (2.8), we get that

$$P_X(s,\chi)^{\ell_\chi} = P_X^*(s,\chi)^{\ell_\chi}\left(1 + O_{\ell_\chi}\left(\frac{1}{\log X}\right)\right).$$

The lemma follows by taking a product over characters χ modulo q. □

2.3 Proof of Theorem 2.1.1

The proof of Theorem 2.1.1 is very similar to [31, Theorem 1] and [16, Theorem 1] so we only provide a sketch of the details.

First, recall that if χ and χ^* are as in the theorem, then

$$L(s,\chi) = L(s,\chi^*) \prod_{\substack{p|q \\ p\nmid q^*(\chi)}} \left(1 - \frac{\chi^*(p)}{p^s}\right). \tag{2.7}$$

Further, by inspection we see that if $P(s,\cdot)$ and $Z(s,\cdot)$ are as in the theorem, then

$$P_X(s,\chi) = P_X(s,\chi^*) \prod_{\substack{p|q \\ p\nmid q^*(\chi)}} \left(1 - \frac{\chi^*(p)}{p^s}\right), \tag{2.8}$$

$$Z_X(s,\chi) = Z_X(s,\chi^*). \tag{2.9}$$

Clearly, (2.7),(2.8) and (2.9) show that we can assume without loss of generality that χ is a primitive character modulo q.

Further, note that we can assume that χ is nonprincipal as, if χ is principal and primitive, the associated L-function is $\zeta(s)$, for which the result was already proved by Gonek, Keating and Hughes [31, Theorem 1].

Our starting point is

$$\log L(s,\chi) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \log n} v(e^{\log n/\log X}) - \sum_{\substack{\rho \\ L(\rho,\chi)=0}} U((s-\rho)\log X), \tag{2.10}$$

which is essentially [16, Equation 8]. Here v and U are as in the theorem, ρ runs over all zeroes of $L(s,\chi)$ including trivial ones, and this representation holds for $\sigma \geqslant 0$, provided s is not a zero of $L(s,\chi)$.

Now, since u is supported on $[e^{1-1/X}, e]$ and is clearly normalized to have mass 1, we can apply the estimates from [31, pp. 515-516]. We thus have

$$\sum_{m=1}^{\infty} U((s+2m)\log X) \ll_{K,f} \frac{X^{K+1}}{(|s|\log X)^K}.$$

Furthermore, by a similar argument, we also have

$$\sum_{m=1}^{\infty} U((s+2m-1)\log X) \ll_{K,f} \frac{X^{K+1}}{(|s|\log X)^K}.$$

Now, since the trivial zeros of $L(s,\chi)$ are either all on negative even integers, or on negative odd integers, upon inserting the above estimates into (2.10), we see that

$$\log L(s_0,\chi) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{s_0}\log n} v(e^{\log n/\log X}) \\ - \sum_{\substack{\rho \\ 0\leqslant \Re\rho\leqslant 1 \\ L(\rho,\chi)=0}} U((s_0-\rho)\log X) \\ + O_{K,f}\left(\frac{X^{K+2}}{(|s_0|\log X)^K}\right).$$

where the sum over ρ now runs only over the non-trivial zeroes of $L(s,\chi)$. Exponentiating, we get

$$L(s,\chi) = \tilde{P}_X(s,\chi) Z_X(s,\chi)\left(1 + O_{K,f}\left(\frac{X^{K+2}}{(|s|\log X)^K}\right)\right),$$

where

$$\tilde{P}_X(s,\chi) = \exp\left(\sum_{n\leqslant X} \frac{\chi(n)\Lambda(n)}{n^s \log n} v(e^{\log n/\log X})\right),$$

and $Z_X(s,\chi)$ is as defined in the theorem.

It remains to replace $\tilde{P}_X(s,\chi)$ by $P_X(s,\chi)$ with a tolerable error and to show that the restriction that $L(s,\chi)\neq 0$ can be removed. This is exactly analogous to [31, pp. 516-517].

2.4 Proof of Theorem 2.1.3

We briefly discuss some notation for this section. Recall that $d_{\boldsymbol{\ell}}(n)$ is the coefficient of n^{-s} in the Dirichlet series of $\mathcal{L}^{\boldsymbol{\ell}}(s)$. $d_{\boldsymbol{\ell}}(n)$ is essentially a divisor function 'twisted' by the Dirichlet characters modulo q. We also use $d_k(n)$ for the true divisor function, i.e., the coefficient of n^{-s} in $\zeta(s)^k$. In particular, it is immediate from writing $d_{\boldsymbol{\ell}}(n)$ out as a convolution that $|d_{\boldsymbol{\ell}}(n)| \leqslant d_k(n)$ for every $n \in \mathbb{N}$. We will use the notation $\mathscr{S}_q(X)$ to denote the set of X-smooth (also known as X-friable) numbers which are coprime to q. That is,

$$\mathscr{S}_q(X) = \{n \in \mathbb{N} : p \mid n \implies p \leqslant X \text{ and } p \nmid q\}.$$

For the rest of this section, we will fix $s = 1/2 + it$. Now, we want to estimate $\int_T^{2T} \left|\mathcal{P}_X^{\boldsymbol{\ell}}(s)\right|^2 dt$ assuming that $q^2 < X \ll_\epsilon (\log T)^{2-\epsilon}$. Clearly, by Lemma 2.2.6,

$$\frac{1}{T}\int_T^{2T} |\mathcal{P}_X^{\boldsymbol{\ell}}(s)|^2\, dt = \left(\frac{1}{T}\int_T^{2T} |\mathcal{P}_X^{*\boldsymbol{\ell}}(s)|^2\, dt\right)\left(1 + O_k\left(\frac{1}{\log X}\right)\right),$$

and so it suffices to compute $\int_T^{2T} |\mathcal{P}_X^{*\boldsymbol{\ell}}(s)|^2\, dt$.

From the definition of $\mathcal{P}_X^{*\boldsymbol{\ell}}(s)$ in Lemma 2.2.6, it follows that if

$$\mathcal{P}_X^{*\boldsymbol{\ell}}(s) = \sum_{n=1}^{\infty} \frac{\beta_{\boldsymbol{\ell}}(n)}{n^s}, \tag{2.11}$$

then $\beta_{\boldsymbol{\ell}}(n)$ is multiplicative and supported on $\mathscr{S}_q(X)$, $|\beta_{\boldsymbol{\ell}}(n)| \leqslant d_{2k}(n)$ for all n, and finally for $n \in \mathscr{S}_q(\sqrt{X})$ and $p \in \mathscr{S}_q(X)$, we have $\beta_{\boldsymbol{\ell}}(n) = d_{\boldsymbol{\ell}}(n)$ and $\beta_{\boldsymbol{\ell}}(p) = d_{\boldsymbol{\ell}}(p)$.

We truncate the sum in (2.11) at T^θ where $\theta > 0$ will be chosen later. Thus,

$$\mathcal{P}_X^{*\boldsymbol{\ell}}(s) = \sum_{\substack{n \in \mathscr{S}_q(X)\\ n \leqslant T^\theta}} \frac{\beta_{\boldsymbol{\ell}}(n)}{n^s} + O\Bigg(\sum_{\substack{n \in \mathscr{S}_q(X)\\ n > T^\theta}} \frac{|\beta_{\boldsymbol{\ell}}(n)|}{n^{1/2}}\Bigg).$$

Applying Rankin's trick and the estimate $|\beta_{\boldsymbol{\ell}}(n)| \leqslant d_{2k}(n)$ to the error term, we see that it is

$$\begin{aligned}\ll_\epsilon \sum_{\substack{n \in \mathscr{S}_q(X)\\ n > T^\theta}} \left(\frac{n}{T^\theta}\right)^\epsilon \frac{|\beta_{\boldsymbol{\ell}}(n)|}{n^{1/2}} \leqslant T^{-\epsilon\theta} &\sum_{n \in \mathscr{S}_q(X)} \frac{d_{2k}(n)}{n^{1/2-\epsilon}} \\ &= T^{-\epsilon\theta} \prod_{\substack{p \leqslant X\\ p \nmid q}} \left(1 - p^{\epsilon - 1/2}\right)^{-2k}.\end{aligned}$$

Using $\log(1-x)^{-1} = O(x)$, we see that the product on the right is

$$T^{-\epsilon\theta} \exp\left(O\left(k \sum_{p \leqslant X} p^{\epsilon - 1/2}\right)\right).$$

Applying the prime number theorem and integrating by parts, we see that since $X \ll_\epsilon (\log T)^{2-\epsilon}$, this is

$$\begin{aligned}
&\ll T^{-\epsilon\theta} \exp\left(O\left(\frac{kX^{1/2+\epsilon}}{(1/2+\epsilon)\log X}\right)\right) \\
&\ll T^{-\epsilon\theta} \exp\left(O_\epsilon\left(\frac{k \log T}{\log\log T}\right)\right) \ll_{k,\epsilon,\theta} T^{-\epsilon\theta/2}.
\end{aligned}$$

Hence, we have

$$\mathcal{P}_X^{*\boldsymbol{\ell}}(s) = \sum_{\substack{n \in \mathscr{S}_q(X) \\ n \leqslant T^\theta}} \frac{\beta_{\boldsymbol{\ell}}(n)}{n^s} + O_{k,\epsilon,\theta}(T^{-\epsilon\theta/2}). \tag{2.12}$$

Now, by the classical mean value theorem for Dirichlet polynomials, we have that

$$\int_T^{2T} \left| \sum_{\substack{n \in \mathscr{S}_q(X) \\ n \leqslant T^\theta}} \frac{\beta_{\boldsymbol{\ell}}(n)}{n^{1/2+it}} \right|^2 dt = (T + O(T^\theta \log T)) \sum_{\substack{n \in \mathscr{S}_q(X) \\ n \leqslant T^\theta}} \frac{|\beta_{\boldsymbol{\ell}}(n)|^2}{n}.$$

Extending the sum on the right hand side to infinity introduces an error $O_{k,\epsilon,\theta}(T^{-\epsilon\theta/2})$, by the same argument as before. Thus, setting $\theta = 1/2$, we see that

$$\frac{1}{T}\int_T^{2T} \left| \sum_{\substack{n \in \mathscr{S}_q(X) \\ n \leqslant T^{1/2}}} \frac{\beta_{\boldsymbol{\ell}}(n)}{n^{1/2+it}} \right|^2 dt = \sum_{n \in \mathscr{S}_q(X)} \frac{|\beta_{\boldsymbol{\ell}}(n)|^2}{n}(1 + O_{k,\epsilon}(T^{-\epsilon/4})) \tag{2.13}$$

Using (2.12) to replace $\mathcal{P}_X^{*\boldsymbol{\ell}}(s)$ with a short Dirichlet polynomial together with (2.13) and Cauchy-Schwarz, we conclude that

$$\frac{1}{T}\int_T^{2T} |\mathcal{P}_X^{*\boldsymbol{\ell}}(s)|^2 \, dt = \sum_{n \in \mathscr{S}_q(X)} \frac{|\beta_{\boldsymbol{\ell}}(n)|^2}{n}(1 + O_{k,\epsilon}(T^{-\epsilon/4})).$$

Thus, it remains to estimate the sum $\sum_{n \in \mathscr{S}_q(X)} \frac{|\beta_{\boldsymbol{\ell}}(n)|^2}{n}$. Since $\beta_{\boldsymbol{\ell}}$ is multiplicative and supported on $\mathscr{S}_q(X)$, we see that

$$\sum_{n\in\mathscr{S}_q(X)}\frac{|\beta_{\boldsymbol{\ell}}(n)|^2}{n}=\prod_{\substack{p\leqslant X\\ p\nmid q}}\left(\sum_{m=0}^{\infty}\frac{|\beta_{\boldsymbol{\ell}}(p^m)|^2}{p^m}\right).$$

Heuristically, $\beta_{\boldsymbol{\ell}}(n)$ was chosen to approximate $d_{\boldsymbol{\ell}}(n)$. So, we expect that we can replace $\beta_{\boldsymbol{\ell}}(p^m)$ with $d_{\boldsymbol{\ell}}(p^m)$ on the right with a tolerable multiplicative error. Now, recall that $\beta_{\boldsymbol{\ell}}(n)=d_{\boldsymbol{\ell}}(n)$ when $n\in\mathscr{S}_q(\sqrt{X})$, and $\beta_{\boldsymbol{\ell}}(p)=d_{\boldsymbol{\ell}}(p)$ for $p\leqslant X$. Thus, we can replace $\beta_{\boldsymbol{\ell}}(p^m)$ by $d_{\boldsymbol{\ell}}(p^m)$ if $p\leqslant\sqrt{X}$ or $m=1$. Hence, it suffices to bound

$$\prod_{\sqrt{X}<p\leqslant X}\frac{1+\frac{|d_{\boldsymbol{\ell}}(p)|^2}{p}+\sum_{m=2}^{\infty}\frac{|\beta_{\boldsymbol{\ell}}(p^m)|^2}{p^m}}{\sum_{m=0}^{\infty}\frac{|d_{\boldsymbol{\ell}}(p^m)|}{p^m}}.$$

However, this is clearly

$$\prod_{\sqrt{X}<p\leqslant X}\left(1+O_k\left(\frac{1}{p^2}\right)\right)=1+O_k\left(\frac{X^{-1/2}}{\log X}\right).$$

Thus,

$$\sum_{n\in\mathscr{S}_q(X)}\frac{|\beta_{\boldsymbol{\ell}}(n)|^2}{n}=\left(1+O_k\left(\frac{X^{-1/2}}{\log X}\right)\right)\prod_{\substack{p\leqslant X\\ p\nmid q}}\left(\sum_{m=0}^{\infty}\frac{|d_{\boldsymbol{\ell}}(p^m)|^2}{p^m}\right). \tag{2.14}$$

Note that we can write the product on the right as

$$\prod_{\substack{p\leqslant X\\ p\nmid q}}\left(\left(1-\frac{1}{p}\right)^{|d_{\boldsymbol{\ell}}(p)|^2}\sum_{m=0}^{\infty}\frac{|d_{\boldsymbol{\ell}}(p^m)|^2}{p^m}\right)\prod_{\substack{p\leqslant X\\ p\nmid q}}\left(1-\frac{1}{p}\right)^{-|d_{\boldsymbol{\ell}}(p)|^2}$$

The constraint $p\leqslant X$ can be removed from the first product here as that induces a multiplicative error given by

$$\begin{aligned}\prod_{p>X}\left(\left(1-\frac{1}{p}\right)^{|d_{\boldsymbol{\ell}}(p)|^2}\sum_{m=0}^{\infty}\frac{|d_{\boldsymbol{\ell}}(p^m)|^2}{p^m}\right)&=\prod_{p>X}\left(1+O_k\left(\frac{1}{p^2}\right)\right)\\&=1+O_k\left(\frac{1}{X\log X}\right).\end{aligned}$$

On doing so, the expression now looks like

$$b(\boldsymbol{\ell})\prod_{\substack{p\leqslant X\\ p\nmid q}}\left(1-\frac{1}{p}\right)^{-|d_{\boldsymbol{\ell}}(p)|^2}. \tag{2.15}$$

Now, define

$$r_\chi = \sum_{\substack{\nu,\nu' \\ \nu\overline{\nu'}=\chi}} \ell_\nu \ell_{\nu'} = \sum_\nu \ell_\nu \ell_{\nu\chi}.$$

In particular, note that $r_\chi = r_{\overline{\chi}}$ and $r_{\chi_0} = \sum_\chi \ell_\chi^2 = \lambda$. Further, define,

$$\kappa(c) = \sum_\chi r_\chi \chi(c).$$

Clearly $\kappa(c)$ is real, and further the definition of $d_{\boldsymbol{\ell}}(n)$ as a convolution gives us that

$$|d_{\boldsymbol{\ell}}(p)|^2 = \sum_\chi r_\chi \chi(p) = \sum_\chi r_\chi \chi(c) = \kappa(c).$$

if $p \equiv c \pmod q$. In particular, this means that the product in (2.15) can be divided along congruence classes modulo q, giving

$$\prod_{(c,q)=1} \prod_{\substack{p \leqslant X \\ p \equiv c \pmod q}} \left(1 - \frac{1}{p}\right)^{-\kappa(c)}.$$

where the outside product runs over a set of representatives of all residue classes coprime to q. Thus, applying Lemma 2.2.1, this is

$$\left(1 + O_q\left(\frac{1}{\log X}\right)\right) \prod_{(c,q)=1} H_c^q(\kappa(c)).$$

In fact, we have that $F_X(\boldsymbol{\ell}) = \prod_{(c,q)=1} H_c^q(\kappa(c))$. To see this, note by orthogonality of characters,

$$\sum_{(c,q)=1} \kappa(c) = \sum_{(c,q)=1} \sum_\chi r_\chi \chi(c) = r_{\chi_0}\varphi(q) = \lambda\varphi(q).$$

Thus,

$$\begin{aligned}
\prod_{(c,q)=1} H_c^q(\kappa(c)) &= \prod_{(c,q)=1} \left[e^\gamma \log X \prod_p \left(1 - \frac{1}{p}\right)^{1-\delta_q(p,c)\varphi(q)} \right]^{\frac{\kappa(c)}{\varphi(q)}} \\
&= (e^\gamma \log X)^\lambda \prod_{(c,q)=1} \prod_p \left(1 - \frac{1}{p}\right)^{\frac{\kappa(c)}{\varphi(q)} - \delta_q(p,c)\kappa(c)} \\
&= (e^\gamma \log X)^\lambda \prod_p \left(1 - \frac{1}{p}\right)^{\lambda - |d_{\boldsymbol{\ell}}(p)|^2} = F_X(\boldsymbol{\ell}).
\end{aligned}$$

Collecting our estimates together proves Theorem 2.1.3.

2.5 Heuristics for Conjecture 2.1.4

We closely follow the arguments in [38, §4] and [31, §4]. We want to heuristically estimate
$$\frac{1}{T}\int_{T}^{2T} |\mathcal{Z}_X^{\boldsymbol{\ell}}(s)|^2\, dt$$
for $s = 1/2 + it$. The factor $Z_X(s,\chi)$ arises as a partial Hadamard product for $L(s,\chi^*)$, where χ^* is the unique primitive character that induces χ. For a fixed χ, $L(s,\chi^*)$ in the t-aspect forms a unitary family, and so we replace each $Z_X(s,\chi)$ with a unitary matrix chosen uniformly with respect to the Haar measure.

The approximate mean density of the zeros of $L(s,\chi^*)$ in the region $0 \leqslant \sigma \leqslant 1$ and $T \leqslant t \leqslant 2T$ is given by
$$\frac{1}{\pi}\mathfrak{D}(\chi,T) = \frac{1}{\pi}\log\left(\frac{q^*(\chi)T}{2\pi}\right)$$
where $q^*(\chi)$ is the conductor of χ. The rescaled zeroes of $L(s,\chi^*)$ at height T are well-modeled by the eigenangles of a uniformly sampled unitary matrix $\mathfrak{U}(N(\chi))$ of size $N(\chi) = \lfloor \mathfrak{D}(\chi,T)\rfloor$.

We now assume the Generalized Riemann Hypothesis for all characters modulo q. Thus, the non-trivial zeros of $L(s,\chi^*)$ are of the form $1/2 + i\gamma(\chi)$ where γ runs over a discrete (multi)set of real numbers depending on χ. Now, consider the trignometric integral
$$\mathrm{Ci}(z) = -\int_z^\infty \frac{\cos w}{w}\, dw.$$
If $E_1(z) = \int_z^\infty e^{-w}w^{-1}\, dw$ is the exponential integral as in Theorem 2.1.1, then $\Re\{E_1(ix)\} = -\mathrm{Ci}(|x|)$.

Hence, using the definition of $\mathcal{Z}_X^{\boldsymbol{\ell}}(s)$ and $Z_X(s,\chi)$,

$$\begin{aligned}\frac{1}{T}\int_T^{2T}\left|\mathcal{Z}_X^{\boldsymbol{\ell}}\left(\tfrac{1}{2}+it\right)\right|^2\,dt &= \frac{1}{T}\int_T^{2T}\prod_{\chi}\left|Z_X\left(\tfrac{1}{2}+it,\chi\right)\right|^{2\ell_\chi}\,dt\\ &= \frac{1}{T}\int_T^{2T}\prod_{\chi}\prod_{\gamma(\chi)}\exp\left(2\ell_\chi\int_1^e u(y)\,\mathrm{Ci}(|t-\gamma(\chi)|\log y\log X)\right)dy\,dt,\end{aligned}$$

where $u(y)$ is a non-negative function of mass 1 supported in $[e^{1-1/X},e]$, as in Theorem 2.1.1, and we have used GRH. Now, following [38, Equation 4.8], if we define $\phi(m,\theta)$ by,

$$\phi(m,\theta)=\exp\left(2m\int_1^e u(y)\,\mathrm{Ci}(|\theta|\log y\log X)\right),$$

then we see that the above integral is modeled by

$$\mathbb{E}\left[\prod_{\chi}\prod_{n=1}^{N(\chi)}\phi(\ell_\chi,\theta_n(\chi))\right],$$

where $\theta_n(\chi)$ is the nth eigenangle of $\mathfrak{U}(N(\chi))$. Here, the expectation is taken against the probability space from which the random matrices $\mathfrak{U}(N(\chi))$ are drawn. In particular, we make an independence assumption between the $\mathfrak{U}(N(\chi))$ for any finite set of distinct characters χ, similar to [38]. Thus, the expectation factorises, giving

$$\prod_{\chi}\mathbb{E}\left[\prod_{n=1}^{N(\chi)}\phi(\ell_\chi,\theta_n(\chi))\right].$$

We can now use [31, Theorem 4] (see also [38, Equation 4.10]), to compute the expectation inside. This gives us

$$\prod_{\chi}\left[\frac{G(\ell_\chi+1)^2}{G(2\ell_\chi+1)}\left(\frac{N(\chi)}{e^\gamma\log X}\right)^{\ell_\chi^2}\left(1+O_{\ell_\chi}\left(\frac{1}{\log X}\right)\right)\right].$$

Finally, recall that $N(\chi)\approx\log(q^*(\chi)T)$, completing the heuristic.

2.6 Proof of Theorem 2.1.6

We begin this section by observing that to prove Theorem 2.1.6 for $|\boldsymbol{\ell}| = 1$, it suffices to verify Conjecture 2.1.4 for $|\boldsymbol{\ell}| = 1$. To see this note that $|\boldsymbol{\ell}| = 1$ is the same as $\boldsymbol{\ell} = \boldsymbol{\delta}^\chi$. Now, it is well-known (see, for example, Lemma 2.2.5) that for a fixed q,

$$\frac{1}{T}\int_T^{2T} \left|L\left(\tfrac{1}{2}+it,\chi\right)\right|^2 dt \sim \frac{\varphi(q)}{q}\log T.$$

Further, putting $\boldsymbol{\ell} = \boldsymbol{\delta}^\chi$ in Theorem 2.1.3 gives

$$\frac{1}{T}\int_T^{2T} \left|P_X\left(\tfrac{1}{2}+it,\chi\right)\right|^2 dt \sim \frac{\varphi(q)}{q}(e^\gamma \log X),$$

provided that $q^2 < X \ll_\epsilon (\log T)^{2-\epsilon}$. Finally, Conjecture 2.1.4 for $\boldsymbol{\ell} = \boldsymbol{\delta}^\chi$ states that for $X, T \to \infty$ with $X \ll_\epsilon (\log T)^{2-\epsilon}$,

$$\frac{1}{T}\int_T^{2T} \left|Z_X\left(\tfrac{1}{2}+it,\chi\right)\right|^2 dt \sim \frac{\log q^*(\chi)T}{e^\gamma \log X}. \tag{2.16}$$

Thus, we see that if we can prove (2.16), then Theorem 2.1.6 follows.

Our first step towards proving (2.16) is the following lemma which is a straightforward corollary of Lemma 2.2.6:

Lemma 2.6.1. *Let $\boldsymbol{\ell}$ be a tuple of nonnegative integers indexed by characters modulo q such that $|\boldsymbol{\ell}| = \sum_\chi \ell_\chi = k$, define*

$$Q_X(s,\chi) = \prod_{p\leqslant\sqrt{X}}\left(1-\frac{\chi(p)}{p^s}\right)\prod_{\sqrt{X}<p\leqslant X}\left(1-\frac{\chi(p)}{p^s}+\frac{\chi(p)^2}{2p^{2s}}\right),$$

and define

$$\mathcal{Q}_X^{\boldsymbol{\ell}}(s) = \prod_\chi Q_X(s,\chi)^{\ell_\chi}.$$

Then, uniformly for $\sigma \geqslant 1/2$ and $X > q^2$,

$$\left[\mathcal{P}_X^{\boldsymbol{\ell}}(s)\right]^{-1} = \mathcal{Q}_X^{\boldsymbol{\ell}}(s)\left(1+O_k\left(\frac{1}{\log X}\right)\right)$$

Proof. Clearly it suffices to restrict ourselves to $\boldsymbol{\ell} = \boldsymbol{\delta}^{\chi}$. Then, by Lemma 2.2.6,

$$\begin{aligned}
P_X(s,\chi)Q_X(s,\chi) &= P_X^*(s,\chi)Q_X(s,\chi)\left(1+O\left(\frac{1}{\log X}\right)\right) \\
&= \left(1+O\left(\frac{1}{\log X}\right)\right)\prod_{\sqrt{X}<p\leqslant X}\left(1+O\left(\frac{1}{p^{3\sigma}}\right)\right) \\
&= 1+O\left(\frac{1}{\log X}\right),
\end{aligned}$$

as desired.

∎

In view of the previous lemma and Theorem 2.1.1, to prove (2.16) we want to show

$$\frac{1}{T}\int_T^{2T}\left|L\left(\tfrac{1}{2}+it,\chi\right)Q_X\left(\tfrac{1}{2}+it,\chi\right)\right|^2\,dt \sim \frac{\log q^*(\chi)T}{e^{\gamma}\log X}.$$

Furthermore, we can assume without loss of generality that χ is primitive. To see this, let χ^* be the Dirichlet character modulo $q^*(\chi)$ which induces χ. Then, $L(s,\chi)$ and $L(s,\chi^*)$ differ only by local factors corresponding to primes p dividing q but not dividing $q^*(\chi)$ and similarly for $X>q^2$, $Q_X(s,\chi)$ and $Q_X(s,\chi^*)$ also differ only by local factors corresponding to such p. In particular, we see that on multiplying these local factors cancel out, giving $L(s,\chi)Q_X(s,\chi) = L(s,\chi^*)Q_X(s,\chi^*)$.

Thus, for χ primitive, we want to show that

$$\frac{1}{T}\int_T^{2T}\left|L\left(\tfrac{1}{2}+it,\chi\right)Q_X\left(\tfrac{1}{2}+it,\chi\right)\right|^2\,dt \sim \frac{\log qT}{e^{\gamma}\log X}.$$

Now, writing $Q_X(s,\chi)$ as a Dirichlet series, we have

$$Q_X\left(s,\chi\right) = \sum_{n=1}^{\infty}\frac{\beta_{-1}(n)}{n^s},$$

where $\beta_{-1}(n)$ is multiplicative and supported on $\mathscr{S}_q(X)$, $|\beta_{-1}(n)| \ll d(n)$, and for $n\in\mathscr{S}_q(\sqrt{X})$ and $p\in\mathscr{S}_q(X)$, we have $\beta_{-1}(n)=\mu(n)\chi(n)$ and $\beta_{-1}(p)=\mu(p)\chi(p)$.

Now, further, define $Q_X(s) = Q_X(s,1)$ where 1 here is the sole character modulo 1, and let

$$Q_X(s) = \sum_{n=1}^{\infty} \frac{\alpha_{-1}(n)}{n^s}.$$

Then we see that $\alpha_{-1}(n)$ as defined above is the same as in [31, §5], and further it is immediate that for $n \in \mathscr{S}_q(X)$, $\beta_{-1}(n) = \alpha_{-1}(n)\chi(n)$.

Mimicking the argument for (2.12), one can show that

$$\begin{aligned} Q_X\left(\tfrac{1}{2}+it,\chi\right) &= \sum_{\substack{n\leqslant T^\theta \\ n\in\mathscr{S}_q(X)}} \frac{\beta_{-1}(n)}{n^{1/2+it}} + O_{\epsilon,\theta}(T^{-\theta\epsilon/10}) \\ &= \sum_{\substack{n\leqslant T^\theta \\ n\in\mathscr{S}_q(X)}} \frac{\alpha_{-1}(n)\chi(n)}{n^{1/2+it}} + O_{\epsilon,\theta}(T^{-\theta\epsilon/10}), \end{aligned} \tag{2.17}$$

for $\epsilon > 0$ small enough.

Putting $\theta = 1/20$, and $b(n) = \alpha_{-1}(n)$ in Lemma 2.2.5, we get that

$$M(T;\chi,\alpha_{-1}) = M'(T;\chi,\alpha_{-1}) + O(T^{-\varepsilon}), \tag{2.18}$$

with $M = M_{\frac{1}{20}}$, $M' = M'_{\frac{1}{20}}$ and $\varepsilon = \varepsilon_{\frac{1}{20}} > 0$.

We first compute the main term $M'(T;\chi,\alpha_{-1})$. Since, $[m,n](m,n) = mn$, $M'(\chi,\alpha_{-1},T)$ is

$$\frac{\varphi(q)}{q} \sum_{\substack{m,n\leqslant T^{1/20} \\ m,n\in\mathscr{S}_q(X)}} \frac{\alpha_{-1}(m)}{m}\frac{\alpha_{-1}(n)}{n}(m,n)\left\{\log\left(\frac{qT(m,n)^2}{2\pi mn}\right) + O_q(1)\right\}.$$

Now, note that any estimates [31, pp. 530-531] can be applied to the above, provided we add the restrictions $(m,q) = (n,q) = (g,q) = 1$ to the sums, and replace $\log T$ with $\log qT$. In particular, following the argument for [31, Equation 34], we conclude that $M'(T;\chi,\alpha_{-1})$ is

$$\frac{\varphi(q)\log qT}{q} \sum_{\substack{m,n\leqslant T^{1/20} \\ m,n\in\mathscr{S}_q(X)}} \frac{\alpha_{-1}(m)}{m}\frac{\alpha_{-1}(n)}{n}(m,n) + O_q((\log X)^{10}).$$

Since $\sum_{\substack{g|m\\ g|n}} \varphi(g) = (m,n)$, the inner sum is

$$\sum_{\substack{m,n\leqslant T^{1/20}\\ m,n\in\mathscr{S}_q(X)}} \frac{\alpha_{-1}(m)}{m}\frac{\alpha_{-1}(n)}{n}\sum_{\substack{g|m\\ g|n}}\varphi(g) = \sum_{\substack{g\leqslant T^{1/20}\\ g\in\mathscr{S}_q(X)}} \frac{\varphi(g)}{g^2}\left(\sum_{\substack{n\leqslant \frac{T^{1/20}}{g}\\ n\in\mathscr{S}_q(X))}} \frac{\alpha_{-1}(gn)}{n}\right)^2$$

Following the argument for [31, Equation 37] here, we can extend both the summations above to infinity to get that $M'(T;\chi,\alpha_{-1})$ is

$$\frac{\varphi(q)\log qT}{q}\sum_{g\in\mathscr{S}_q(X)}\frac{\varphi(g)}{g^2}\left(\sum_{n\in\mathscr{S}_q(X))}\frac{\alpha_{-1}(gn)}{n}\right)^2 + O_q((\log X)^{10}).$$

By the muliplicativity of α_{-1} and φ, the sum here can be written as an Euler product

$$\prod_{\substack{p\leqslant X\\ p\nmid q}}\left(\sum_{r,j,k\geqslant 0}\frac{\varphi(p^r)\alpha_{-1}(p^{r+j})\alpha_{-1}(p^{r+k})}{p^{2r+j+k}}\right).$$

Now, recalling that $\alpha_{-1}(n) = \mu(n)$ if $n \in \mathscr{S}_q(\sqrt{X})$, $\alpha_{-1}(p) = \mu(p)$ for all $p \leqslant X$ and $\alpha_{-1}(n) \ll d(n)$ for all $n \in \mathscr{S}_q(X)$, we get that this product is equal to

$$\begin{aligned}
&\prod_{\substack{p\leqslant\sqrt{X}\\ p\nmid q}}\left(1-\frac{1}{p}\right)\prod_{\substack{\sqrt{X}<p\leqslant X\\ p\nmid q}}\left(1-\frac{1}{p}+O\left(\frac{1}{p^2}\right)\right)\\
&\qquad = \frac{q}{\varphi(q)}\prod_{p\leqslant X}\left(1-\frac{1}{p}\right)\prod_{\sqrt{X}<p\leqslant X}\left(1+O\left(\frac{1}{p^2}\right)\right)\\
&\qquad = \frac{q}{\varphi(q)}\cdot\frac{1}{e^\gamma\log X}\left(1+O\left(\frac{1}{\log X}\right)\right).
\end{aligned}$$

Thus, since $\log X \ll \log\log T$, we see that, in fact

$$M'(T;\chi,\alpha_{-1}) = \frac{\log qT}{e^\gamma\log X}\left(1+O\left(\frac{1}{\log X}\right)\right) \tag{2.19}$$

Writing (2.17) with $\theta = 1/20$ as $Q_X(1/2+it,\chi) = Q_X^* + O(T^{-\epsilon/200})$,

$$\frac{1}{T}\int_T^{2T}\left|L\left(\tfrac{1}{2}+it,\chi\right)Q_X\left(\tfrac{1}{2}+it,\chi\right)\right|^2\,dt$$
$$=\frac{1}{T}\int_T^{2T}\left|L\left(\tfrac{1}{2}+it,\chi\right)Q_X^*\right|^2\,dt$$
$$+\,O\left(\frac{1}{T^{1+\frac{\epsilon}{200}}}\int_T^{2T}\left|L\left(\tfrac{1}{2}+it,\chi\right)^2Q_X^*\right|\,dt\right)$$
$$+\,O\left(\frac{1}{T^{1+\frac{\epsilon}{100}}}\int_T^{2T}\left|L\left(\tfrac{1}{2}+it,\chi\right)\right|^2\,dt\right).$$

The first term here is $M'(T;\chi,\alpha_{-1})+O(T^{-\varepsilon})$. The last term is $\ll_q T^{-\epsilon/200}$ since the second moment of $L(s,\chi)$ is $\ll_q T\log T$. Finally, by Cauchy-Schwarz and (2.19), the second term is

$$\ll \frac{1}{T^{1+\frac{\epsilon}{200}}}\left(\int_T^{2T}\left|L\left(\tfrac{1}{2}+it,\chi\right)Q_X^*\right|^2\,dt\int_T^{2T}\left|L\left(\tfrac{1}{2}+it,\chi\right)\right|^2\,dt\right)^{1/2}$$
$$\ll \frac{1}{T^{1+\frac{\epsilon}{200}}}\left(\frac{T^2(\log T)^2}{\log X}\right)^{1/2}\ll T^{-\frac{\epsilon}{400}}.$$

Putting these estimates together with (2.19), we get that

$$\frac{1}{T}\int_T^{2T}\left|L\left(\tfrac{1}{2}+it,\chi\right)Q_X\left(\tfrac{1}{2}+it,\chi\right)\right|^2\,dt$$
$$=M'(T;\chi,\alpha_{-1})+O(T^{-\vartheta})$$
$$=\frac{\log qT}{e^\gamma\log X}\left(1+O\left(\frac{1}{\log X}\right)\right),$$

for some $\vartheta=\vartheta(\epsilon,\varepsilon_{\frac{1}{20}})>0$ completing the proof of Theorem 2.1.6.

2.7 Proof of Theorem 2.1.5

In this section, we prove Theorem 2.1.5, and verify that the conjectural constant $c_{\boldsymbol{\ell}}(q)$ matches up with the constants in the known asymptotics (see Lemmata 2.2.3 and 2.2.4).

The proof is straightforward. By Theorem 2.1.3, we get that assuming we hold $q, \boldsymbol{\ell}$, and ϵ fixed, and let $X, T \to \infty$ with $X \ll_\epsilon (\log T)^{2-\epsilon}$,

$$\frac{1}{T}\int_T^{2T} |\mathcal{P}_X^{\boldsymbol{\ell}}(s)|^2\, dt = (e^\gamma \log X)^\lambda \prod_p \left\{ \left(1 - \frac{1}{p}\right)^\lambda \sum_{m=0}^\infty \frac{|d_{\boldsymbol{\ell}}(p^m)|^2}{p^m} \right\}. \tag{2.20}$$

Further, since we are assuming Conjecture 2.1.4 for $\boldsymbol{\ell}$, we get that under the same conditions as before,

$$\begin{aligned} \frac{1}{T}\int_T^{2T} |\mathcal{Z}_X^{\boldsymbol{\ell}}(s)|^2\, dt &\sim \prod_\chi \left[\frac{G(\ell_\chi+1)^2}{G(2\ell_\chi+1)} \left(\frac{\log q^*(\chi)T}{e^\gamma \log X} \right)^{\ell_\chi^2} \right] \\ &= \frac{1}{(e^\gamma \log X)^\lambda} \prod_\chi \left[\frac{G(\ell_\chi+1)^2}{G(2\ell_\chi+1)} (\log q^*(\chi)T)^{\ell_\chi^2} \right]. \end{aligned} \tag{2.21}$$

Finally, since we are assuming that Conjecture 2.1.2 is true for $\boldsymbol{\ell}$, we get that for X, T as before,

$$\frac{1}{T}\int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}(s)\right|^2\, dt \sim \left(\frac{1}{T}\int_T^{2T} \left|\mathcal{P}_X^{\boldsymbol{\ell}}(s)\right|^2\, dt \right) \times \left(\frac{1}{T}\int_T^{2T} \left|\mathcal{Z}_X^{\boldsymbol{\ell}}(s)\right|^2\, dt \right).$$

Multiplying (2.20) and (2.21) and inserting above, we see that the $(e^\gamma \log X)^\lambda$ factors cancel out, and the constants combine to become $c_{\boldsymbol{\ell}}(q)$, giving

$$\frac{1}{T}\int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}(s)\right|^2\, dt \sim c_{\boldsymbol{\ell}}(q) \prod_\chi \left(\log q^*(\chi)T \right)^{\ell_\chi^2},$$

as desired.

We now verify that the constants match up. Let $C'(\chi)$ and $D'(\chi, \nu)$ be the constants predicted by Theorem 2.1.5. Then, $C'(\chi) = c_{\boldsymbol{\ell}}(q)$ for $\boldsymbol{\ell} = 2\boldsymbol{\delta}^\chi$, and $D'(\chi, \nu) = c_{\boldsymbol{\ell}}(q)$ for $\boldsymbol{\ell} = \boldsymbol{\delta}^\chi + \boldsymbol{\delta}^\nu$, $\chi \neq \nu$.

To show that $C(\chi) = C'(\chi)$ and $D(\chi, \nu) = D'(\chi, \nu)$, the plan of attack will be to write everything involved as an Euler product, and then compare what happens on both sides in the local factors for different primes p.

In particular, recall Ingham's result that $c_2 = \frac{1}{2\pi^2}$. Thus, using this, we can suppress the local factors for $p \nmid q$ when showing $C(\chi) = C'(\chi)$. Rewriting $C(\chi)$

in Euler product form using a standard formula for $\varphi(q)/q$, we see that

$$\begin{aligned} C(\chi) &= c_2 \prod_{p \mid q} \left(1 - \frac{1}{p}\right)^2 \left(1 - \frac{2}{p+1}\right) \\ &= c_2 \prod_{p \mid q} \left(1 - \frac{1}{p}\right)^3 \left(1 + \frac{1}{p}\right)^{-1}. \end{aligned} \tag{2.22}$$

Now since $C'(\chi) = c_{\boldsymbol{\ell}}(q)$ for $\boldsymbol{\ell} = 2\boldsymbol{\delta}^{\chi}$, we get that $\lambda = 2^2 = 4$, $d_{\boldsymbol{\ell}}(n) = \chi_0(n) d_2(n)$ where χ_0 is the principal character modulo q, and hence

$$C'(\chi) = c_{\boldsymbol{\ell}}(q) = \left[\prod_p \left\{\left(1 - \frac{1}{p}\right)^4 \sum_{m=0}^{\infty} \frac{\chi_0(p^m) d_2(p^m)^2}{p^m}\right\}\right] \left[\frac{G(3)^2}{G(5)}\right].$$

Recall that

$$c_2 = \left[\prod_p \left\{\left(1 - \frac{1}{p}\right)^4 \sum_{m=0}^{\infty} \frac{d_2(p^m)^2}{p^m}\right\}\right] \left[\frac{G(3)^2}{G(5)}\right].$$

Thus, we see that

$$C'(\chi) = c_2 \prod_{p \mid q} \left\{\sum_{m=0}^{\infty} \frac{d_2(p^m)^2}{p^m}\right\}^{-1}. \tag{2.23}$$

In light of (2.22) and (2.23), it suffices to note the power series equality

$$\sum_{m=0}^{\infty} d_2(p^m)^2 z^m = \frac{1+z}{(1-z)^3},$$

for $|z| < 1$, as then plugging in $z = 1/p$ and taking products over $p \mid q$ gives us $C(\chi) = C'(\chi)$. To see the above power series equality, note that $d_2(p^m) = m+1$ and hence this follows straightforwardly from the geometric series formula.

Now we turn to showing $D(\chi, \nu) = D'(\chi, \nu)$. Since $D'(\chi, \nu) = c_{\boldsymbol{\ell}}(q)$ for $\boldsymbol{\ell} = \boldsymbol{\delta}^{\chi} + \boldsymbol{\delta}^{\nu}$, hence $\mathcal{L}^{\boldsymbol{\ell}}(s) = L(s, \chi) L(s, \nu)$ and $d_{\boldsymbol{\ell}} = \chi * \nu$, where $*$ denotes Dirichlet convolution. Because ν is completely multiplicative, $d_{\boldsymbol{\ell}}(n) = \nu(n)\{1 * (\chi\overline{\nu})\}(n)$. In particular, it follows that $|d_{\boldsymbol{\ell}}(n)|^2$ depends only on $\chi\overline{\nu}$ and not the individual characters χ and ν. Thus, $D'(\chi, \nu)$ also depends only on $\chi\overline{\nu}$. By inspection, we see that $D(\chi, \nu)$ also depends only on $\chi\overline{\nu}$. Thus, without loss of generality, we can assume that $\nu = \chi_0$. It now suffices to show that for $\chi \neq \chi_0$, $D(\chi, \chi_0) = D'(\chi, \chi_0)$.

For $\boldsymbol{\ell} = \boldsymbol{\delta}^{\chi} + \boldsymbol{\delta}^{\chi_0}$, we see that $\lambda = 1^2 + 1^2 = 2$. Further, $d_{\boldsymbol{\ell}}(n) = \chi_0(n)\{1 * \chi\}(n)$. Finally the product over χ in the expression for $c_{\boldsymbol{\ell}}(q)$ vanishes, since $G(1)^2/G(3) = 1$. Thus, we get

$$D'(\chi, \chi_0) = \prod_p \left\{ \left(1 - \frac{1}{p}\right)^2 \sum_{m=0}^{\infty} \frac{\chi_0(p^m)|(1 * \chi)(p^m)|^2}{p^m} \right\} \tag{2.24}$$

Now, using the Euler product formulae,

$$\frac{6}{\pi^2} = \frac{1}{\zeta(2)} = \prod_p \left(1 - \frac{1}{p^2}\right)$$

and

$$L(1, \chi) = \prod_p \frac{1}{1 - \chi(p)p^{-1}},$$

where the latter holds because $\chi \neq \chi_0$, we see that

$$\begin{aligned} D(\chi, \chi_0) &= \frac{6}{\pi^2} |L(1, \chi)|^2 \frac{\varphi(q)}{q} \prod_p \left(1 - \frac{1}{p+1}\right) \\ &= \left\{ \prod_p \frac{1 - p^{-2}}{(1 - \chi(p)p^{-1})(1 - \overline{\chi}(p)p^{-1})} \right\} \left\{ \prod_{p|q} \frac{1 - p^{-1}}{1 + p^{-1}} \right\}. \end{aligned} \tag{2.25}$$

Comparing the local factors corresponding to primes p dividing q, we see that for $D'(\chi, \chi_0)$ these are $(1 - p^{-1})^2$, while for $D(\chi, \chi_0)$, they are

$$\frac{(1 - p^{-2})(1 - p^{-1})}{1 + p^{-1}} = (1 - p^{-1})^2.$$

Thus, it remains to check the local factors corresponding to primes p which are coprime to q. For $D(\chi, \chi_0)$, these are of the shape

$$\frac{1 - p^{-2}}{(1 - \chi(p)p^{-1})(1 - \overline{\chi}(p)p^{-1})},$$

while for $D'(\chi, \chi_0)$, these are of the shape

$$\left(1 - \frac{1}{p}\right)^2 \sum_{m=0}^{\infty} \frac{|(1 * \chi)(p^m)|^2}{p^m}.$$

Thus, to prove $D(\chi,\chi_0) = D'(\chi,\chi_0)$ it clearly suffices to prove the power series equality

$$\frac{1+z}{(1-\omega z)(1-\overline{\omega} z)} = (1-z)\sum_{m=0}^{\infty}\left|\sum_{j=0}^{m}\omega^j\right|^2 z^m,$$

for $|z|<1$ and $|\omega|=1$, as then plugging in $z=1/p$, $\omega=\chi(p)$ and multiplying both sides by $(1-p^{-1})$ gives us the desired equality.

To prove this power series equality note that both sides are equal to

$$\sum_{m\geqslant 0}\left(\sum_{|j|\leqslant m}\omega^j\right) z^m, \tag{2.26}$$

where the sum over j runs through all integers in $[-m,m]$. For the right hand side, this follows from opening the square; for the left hand side it follows from the geometric series formula.

This discussion shows that the conjectural constants $c_{\boldsymbol{\ell}}(q)$ from Theorem 2.1.5 are correct for $\boldsymbol{\ell} = \boldsymbol{\delta}^{\chi} + \boldsymbol{\delta}^{\nu}$ where χ,ν are not necessarily distinct Dirichlet characters modulo q. One could, in principle, use Topacogullari's results from [82] to verify the analoguous constants for products of the form $L(s,\chi)L(s,\nu)$ with χ,ν possibly having distinct moduli.

3 Moments of the Hurwitz zeta function with rational shifts

3.1 Introduction

Our goal in this chapter is to present the evidence for Conjecture 1.0.1, making use of the results of Chapter 2 when necessary. The conjectural asymptotic

$$M_k(T;\alpha) \sim c_k(\alpha)T(\log T)^{k^2},$$

for *some* constant $c_k(\alpha)$ given $\alpha \in \mathbb{Q}$ is not hard to believe given (1.1). Our particular aim here is to explore the structure of $c_k(\alpha)$.

When $k = 1$, Conjecture 1.0.1 is actually a theorem due to Rane [73, Theorem 2], with $c_1(\alpha) = 1$. In fact, he proved for $0 < \alpha \leqslant 1$ (not necessarily rational),

$$M_1(T;\alpha) = T\log T + B(\alpha)T - \frac{1}{\alpha} + O\left(\frac{T^{1/2}\log T}{\alpha^{1/2}}\right) \tag{3.1}$$

uniformly in α and T with an effective constant $B(\alpha)$. This was improved further by several authors, with the current best error term due to Zhan [89, Theorem 2].

For $k = 2$, the conjecture can be proved using methods for fourth moments of L-functions of degree 1. This was done in an unpublished section of Andersson's thesis [1, pp. 71-72]. We restate and reprove this result here for convenience:

Theorem 3.1.1. *Let $a, q \geqslant 1$ be fixed integers with $(a,q) = 1$, $1 \leqslant a \leqslant q$. Then, for $\alpha = a/q$,*

$$M_2(T;\alpha) = \int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it,\alpha\right)\right|^4 \, dt \sim \frac{T(\log T)^4}{2\pi^2 q} \prod_{p|q}\left(1 - \frac{1}{p+1}\right),$$

as $T \to \infty$. That is, Conjecture 1.0.1 is true for $k = 2$ and $\alpha = a/q$, with

$$c_2(\alpha) = \frac{1}{2\pi^2 q} \prod_{p|q}\left(1 - \frac{1}{p+1}\right) = \frac{c_2}{q} \prod_{p|q}\left(1 - \frac{1}{p+1}\right),$$

where $c_2 = c_2(1) = 1/(2\pi^2)$ is the usual proportionality constant for the fourth moment of $\zeta(s)$. More precisely, we have

$$M_2(T;\alpha) = c_2(\alpha) T(\log T)^4 + O_q(T(\log T)^3).$$

We show later that this agrees with our conjecture for $c_k(\alpha)$. In principle, one could also work out the lower order terms in this asymptotic.

It should be evident that our previous discussions about (2.2) and Theorem 2.1.5 can together be used to compute the correct value of $c_k(\alpha)$ in Conjecture 1.0.1.

Theorem 3.1.2. *Let $k \geqslant 0$ and $a, q \geqslant 1$ be fixed integers with $(a,q) = 1$, $1 \leqslant a \leqslant q$. If Conjecture 2.1.2 and Conjecture 2.1.4 are true for all tuples of nonnegative integers $\boldsymbol{\ell}$ indexed by characters modulo q satisfying $|\boldsymbol{\ell}| = k$, then Conjecture 1.0.1 follows for that value of k and $\alpha = a/q$. In other words, under the above hypotheses,*

$$\int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it,\alpha\right)\right|^{2k} \, dt \sim c_k(\alpha) T(\log T)^{k^2},$$

as $T \to \infty$ where $c_k(\alpha)$ is as in (1.4).

Note that Theorem 3.1.2 and Theorem 2.1.6 together establish Conjecture 1.0.1 with $k = 1$ and α rational, giving an alternate proof of the leading term of Rane's asymptotic (3.1) in this case.

Lastly, as a final piece of evidence for Conjecture 1.0.1, we prove the following results about upper and lower bounds:

Theorem 3.1.3. *Let $k \geqslant 0$ and $a, q \geqslant 1$ be fixed integers with $(a, q) = 1$, $1 \leqslant a \leqslant q$. If the Generalized Riemann Hypothesis (GRH) holds for every Dirichlet L-function modulo q, then for $\alpha = a/q$, $s = 1/2 + it$ and $\epsilon > 0$,*

$$T(\log T)^{k^2} \ll_{q,k} \int_T^{2T} |\zeta(s, \alpha)|^{2k}\, dt \ll_{q,k,\epsilon} T(\log T)^{k^2+\epsilon}$$

In principle, it should be possible to remove the ϵ in the upper bound by using the methods of Harper [34].

3.2 Proof of Theorem 3.1.2

The main result of this section is the following proposition which is one way to make the heuristic in (2.2) rigorous:

Proposition 3.2.1. *Let $M_k(T; \alpha)$ be as in (1.3), and for any Dirichlet character χ modulo q, define $M_k(T; \chi)$ by*

$$M_k(T; \chi) = \int_T^{2T} \left|L\left(\tfrac{1}{2} + it, \chi\right)\right|^{2k} dt.$$

If either $\mathsf{GRH}(q)$ or $\mathsf{Sp}(q, k)$ holds, and $\alpha = a/q$ with $(a, q) = 1$, then

$$M_k(T; \alpha) = \frac{q^k}{\varphi(q)^{2k}} \sum_{\chi} M_k(T; \chi) + o_{q,k}(T(\log T)^{k^2}).$$

If $k = 1$ or $k = 2$, then the above can be proved unconditionally.

We show that this proposition establishes Theorem 3.1.2. Note that under the hypothesis of Theorem 3.1.2, $\mathsf{Sp}(q, k)$ holds, and hence so does the conclusion of Theorem 2.1.5. Thus, for a fixed q, χ, and with $\boldsymbol{\ell} = k\boldsymbol{\delta}^{\chi}$, we get the asymptotic

$$M_k(T; \chi) = (c_{\boldsymbol{\ell}}(q) + o_{q,k}(1))T(\log T)^{k^2}.$$

Thus, by Proposition 3.2.1,

$$M_k(T;\alpha) = \frac{q^k}{\varphi(q)^{2k}} \left(\sum_{\boldsymbol{\ell} = k\boldsymbol{\delta}^{\chi}} c_{\boldsymbol{\ell}}(q) \right) T(\log T)^{k^2} + o_{q,k}(T(\log T)^{k^2}),$$

which establishes Theorem 3.1.2 with

$$c_k(\alpha) = \frac{q^k}{\varphi(q)^{2k}} \left(\sum_{\boldsymbol{\ell} = k\boldsymbol{\delta}^{\chi}} c_{\boldsymbol{\ell}}(q) \right), \tag{3.2}$$

where the sum runs over all tuples $\boldsymbol{\ell}$ of the form $k\boldsymbol{\delta}^{\chi}$ for some character χ. It remains to simplify the constant. Note that for $\boldsymbol{\ell} = k\boldsymbol{\delta}^{\chi}$, $d_{\boldsymbol{\ell}}(n) = \chi(n)d_k(n)$, where $d_k(n)$ is the usual divisor function. In particular, this means that $|d_{\boldsymbol{\ell}}(p^m)|^2 = \chi_0(p^m)d_k(p^m)^2$, and hence $c_{\boldsymbol{\ell}}(q)$ depends only on the modulus of χ. Further, $\lambda(\boldsymbol{\ell}) = k^2$. Thus,

$$c_{\boldsymbol{\ell}}(q) = \prod_p \left\{ \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \frac{\chi_0(p^m)d_k(p^m)^2}{p^m} \right\} \frac{G(k+1)}{G(2k+1)},$$

for every $\boldsymbol{\ell}$ appearing in the sum in (3.2). We see that $c_{\boldsymbol{\ell}}(q)$ is the same as usual constant for $\zeta(s)$, $c_k = c_k(1)$ but with a slight change in the local factors in the Euler product corresponding to those primes p which divide q. That is,

$$c_{\boldsymbol{\ell}}(q) = c_k \prod_{p|q} \left\{ \sum_{m=0}^{\infty} \frac{d(p^m)^2}{p^m} \right\}^{-1} = c_k \prod_{p|q} \left\{ \sum_{m=0}^{\infty} \binom{m+k-1}{k-1}^2 p^{-m} \right\}^{-1}. \tag{3.3}$$

Substituting this back into (3.2),

$$c_k(\alpha) = c_k \frac{q^k}{\varphi(q)^{2k-1}} \prod_{p|q} \left\{ \sum_{m=0}^{\infty} \binom{m+k-1}{k-1}^2 p^{-m} \right\}^{-1},$$

as desired. This completes the proof of Theorem 3.1.2 from Proposition 3.2.1.

We now turn to the proof of Proposition 3.2.1. By (2.1), $M_k(T;\alpha)$ is equal to

$$\frac{q^k}{\varphi(q)^{2k}} \sum_{\substack{|\boldsymbol{\ell}^{(1)}|=k, \\ |\boldsymbol{\ell}^{(2)}|=k}} \binom{k}{\boldsymbol{\ell}^{(1)}} \binom{k}{\boldsymbol{\ell}^{(2)}} \left[\prod_{\chi} \chi(a)^{\ell_{\chi}^{(2)} - \ell_{\chi}^{(1)}} \right] \int_T^{2T} \mathcal{L}^{\boldsymbol{\ell}^{(1)}}(s) \overline{\mathcal{L}^{\boldsymbol{\ell}^{(2)}}(s)}\, dt, \tag{3.4}$$

where $s = 1/2 + it$. We divide the terms in the sum into four types:

- The primary diagonal terms. These correspond to $\boldsymbol{\ell}^{(1)} = \boldsymbol{\ell}^{(2)} = k\boldsymbol{\delta}^{\chi}$ for some character χ. For such terms, it is clear that $\binom{k}{\boldsymbol{\ell}^{(j)}} = 1$ and the integral devolves to $M_k(T;\chi)$.

- The secondary diagonal terms. These correspond to diagonal terms $\boldsymbol{\ell}^{(1)} = \boldsymbol{\ell}^{(2)}$ which are not main diagonal terms. Thus, $\boldsymbol{\ell} = \boldsymbol{\ell}^{(1)} = \boldsymbol{\ell}^{(2)} \neq k\boldsymbol{\delta}^{\chi}$ for every character χ modulo q. For such terms, the integral devolves to the mean square of $\mathcal{L}^{\boldsymbol{\ell}}(1/2+it)$ over $[T, 2T]$.

- The major off-diagonal terms. These correspond to $\boldsymbol{\ell}^{(1)} = k\boldsymbol{\delta}^{\chi}$ and $\boldsymbol{\ell}^{(2)} = k\boldsymbol{\delta}^{\nu}$ for distinct characters χ, ν. For these terms, the integral devolves to $\int_T^{2T} L(s,\chi)^k \overline{L(s,\nu)}^k \, dt$.

- The minor off-diagonal terms. These correspond to any terms which are not of any of the above three forms.

The primary diagonal terms clearly give rise to the main term in Proposition 3.2.1. We will show, through a series of lemmata, that all the other terms can be subsumed by the error term, thereby proving the proposition.

Lemma 3.2.2. *The secondary diagonal terms contribute at most $o_{q,k}(T(\log T)^{k^2})$ to the sum in (3.4).*

Proof. We simply apply Lemma 2.2.2. It is clear that the constant $\binom{k}{\boldsymbol{\ell}}^2$ in the the secondary diagonal terms can be subsumed into the implicit constant from the second bound in this lemma. Taking, for example, $\epsilon = 1/2$ shows that each such term is $o_{q,k}(T(\log T)^{k^2})$ and since there are only $\ll_{q,k} 1$ such terms it follows that these contribute only $o_{q,k}(T(\log T)^{k^2})$. □

The minor off-diagonal terms can be handled by Cauchy-Schwarz. This is very far from sharp, but our other error terms are already of size $\asymp_{q,k} T(\log T)^{k^2-2k+2}$ and so this suffices for our purposes.

Lemma 3.2.3. *Suppose that either* $\mathsf{GRH}(q)$ *or* $\mathsf{Sp}(q,k)$ *holds and that* $\boldsymbol{\ell}^{(1)}$ *and* $\boldsymbol{\ell}^{(2)}$ *are tuples of nonnegative integers characters modulo* q *satisfying* $|\boldsymbol{\ell}^{(1)}| = |\boldsymbol{\ell}^{(2)}| = k$. *Further, suppose that* $\boldsymbol{\ell}^{(1)}, \boldsymbol{\ell}^{(2)}$ *correspond to a minor off-diagonal term, as defined above. Then, for* $s = 1/2 + it$ *and any* $\epsilon > 0$,

$$\int_T^{2T} \mathcal{L}^{\boldsymbol{\ell}^{(1)}}(s)\overline{\mathcal{L}^{\boldsymbol{\ell}^{(2)}}(s)}\, dt \ll_{q,k,\epsilon} T(\log T)^{k^2-1/2+\epsilon},$$

and hence, the minor off-diagonal terms in (3.4) contribute at most $o_{q,k}(T(\log T)^{k^2})$ *to the sum.*

Proof. Since $(\boldsymbol{\ell}^{(1)}, \boldsymbol{\ell}^{(2)})$ corresponds to an off-diagonal term, $\boldsymbol{\ell}^{(1)} \neq \boldsymbol{\ell}^{(2)}$. Further, since it is not a major off-diagonal term we must have that either $\boldsymbol{\ell}^{(1)}$ or $\boldsymbol{\ell}^{(2)}$ is not of the form $k\boldsymbol{\delta}^{\chi}$ for some character χ. Due to symmetry, we can assume without loss of generality that $\boldsymbol{\ell}^{(1)} \neq k\boldsymbol{\delta}^{\chi}$ for all characters χ. Then, by Cauchy-Schwarz and Lemma 2.2.2, we get for $s = 1/2 + it$,

$$\begin{aligned}
&\int_T^{2T} \mathcal{L}^{\boldsymbol{\ell}^{(1)}}(s)\overline{\mathcal{L}^{\boldsymbol{\ell}^{(2)}}(s)}\, dt \\
&\qquad \ll \left(\int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}^{(1)}}(s)\right|^2 dt\right)^{1/2} \left(\int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}^{(2)}}(s)\right|^2 dt\right)^{1/2} \\
&\qquad \ll_{q,k,\epsilon} \left\{T(\log T)^{k^2-1+\epsilon}\right\}^{1/2} \left\{T(\log T)^{k^2+\epsilon}\right\}^{1/2} \\
&\qquad = T(\log T)^{k^2-1/2+\epsilon}.
\end{aligned}$$

Showing that these terms contribute to the error is similar to the previous lemma, hence we omit the proof.

□

We note in passing that if $k = 1$, there are no secondary diagonal terms or minor off-diagonal terms, and so the previous two lemmata are unnecessary.

It remains to deal with the major non-diagonal terms. If $k \geqslant 2$, then again Cauchy-Schwarz suffices.

Lemma 3.2.4. *Suppose that either* $\mathsf{GRH}(q)$ *or* $\mathsf{Sp}(q,k)$ *holds for some* $k \geqslant 2$ *and that* χ *and* ν *are distinct characters modulo* q*. Then, for* $s = 1/2 + it$ *and any* $\epsilon > 0$*,*

$$\int_T^{2T} L(s,\chi)^k \overline{L(s,\nu)}^k \, dt \ll_{q,k,\epsilon} T(\log T)^{k^2-2k+2+\epsilon},$$

and hence, the major off-diagonal terms in (3.4) contribute at most $o_{q,k}(T(\log T)^{k^2})$ *to the sum.*

Proof. Note that,

$$L(s,\chi)^k \overline{L(s,\nu)}^k = \left[L(s,\chi)^{k-1}\overline{L(s,\nu)}\right]\left[L(s,\chi)\overline{L(s,\nu)}^{k-1}\right].$$

Thus, setting

$$\boldsymbol{\ell}^{(1)} = (k-1)\boldsymbol{\delta}^{\chi} + \boldsymbol{\delta}^{\nu}$$

and

$$\boldsymbol{\ell}^{(2)} = \boldsymbol{\delta}^{\chi} + (k-1)\boldsymbol{\delta}^{\nu},$$

we see by Cauchy-Schwarz and Lemma 2.2.2 that

$$\begin{aligned}
&\int_T^{2T} L(s,\chi)^k \overline{L(s,\nu)}^k \, dt \\
&\qquad \ll \left(\int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}^{(1)}}(1/2+it)\right|^2 dt\right)^{1/2} \left(\int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}^{(2)}}(1/2+it)\right|^2 dt\right)^{1/2} \\
&\qquad \ll_{q,k,\epsilon} \left\{T(\log T)^{k^2-2k+2+\epsilon}\right\}^{1/2} \left\{T(\log T)^{k^2-2k+2+\epsilon}\right\}^{1/2} \\
&\qquad = T(\log T)^{k^2-2k+2+\epsilon},
\end{aligned}$$

proving the desired bound. Showing that these terms contribute to the error is similar to the above lemmata, and hence omitted.

$\square$

From the discussion above, it remains to deal with the off-diagonal terms when $k = 1$, and to show that the argument can be made unconditional for $k = 2$.

We postpone the latter to §3.3, as it will be a corollary of the discussion about Theorem 3.1.1.

For the former, since we also claimed that Proposition 3.2.1 is unconditional in this case, we cannot use the hypotheses $\mathsf{GRH}(q)$ or $\mathsf{Sp}(q,k)$. For such terms, standard techniques developed to handle the mean square of $\zeta(s)$ can be applied. For our purposes, the following lemma suffices:

Lemma 3.2.5. *Let χ and ν be distinct characters modulo q. Then, for $s = 1/2+it$,*

$$\int_T^{2T} L(s,\chi)\overline{L(s,\nu)}\,dt \ll_q T(\log T)^{3/4},$$

unconditionally. Hence, if $k=1$, the off-diagonal terms in (3.4) contribute only $o_q(T\log T)$ to the sum.

Proof. The upper bound is [50, Equation 4] with $\chi_j = \chi$ and $\chi_k = \nu$. Showing that these terms contribute to the error is similar to the previous lemmata, and hence omitted.

$\square$

Proposition 3.2.1 follows by putting all these lemmata together, thus completing the proof of Theorem 3.1.2.

3.2.1 Proof of Theorem 3.1.3

In order to prove Theorem 3.1.3, we have to find bounds on $M_k(T;\alpha)$ conditionally on GRH.

The claimed upper bound follows trivially from the previous subsection, since Proposition 3.2.1 tells us that on $\mathsf{GRH}(q)$,

$$M_k(T;\alpha) \ll_{q,k} \sum_{\chi} M_k(T;\chi) + T(\log T)^{k^2},$$

and Lemma 2.2.2 tells us that on $\mathsf{GRH}(q)$,

$$M_k(T;\chi) \ll_{q,k,\epsilon} T(\log T)^{k^2+\epsilon}.$$

To prove the lower bound, we proceed by reducing the problem to computing lower bounds for the moments of $\zeta(s)$, i.e. lower bounds on $M_k(T)$. The key fact is the following obvious lemma:

Lemma 3.2.6. *Let χ_0 be the principal Dirichlet character modulo q. Then,*

$$\int_T^{2T} \left|L\left(\tfrac{1}{2}+it,\chi_0\right)\right|^{2k} dt \asymp_{q,k} \int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt.$$

In particular, this tells us that $M_k(T;\chi_0) \gg_{q,k} M_k(T)$. By the deep results in the literature about lower bounds for $M_k(T)$ mentioned in the introduction, we can conclude that in fact $M_k(T;\chi_0) \gg_{q,k} T(\log T)^{k^2}$.

Then, by Proposition 3.2.1, we have conditionally on GRH,

$$\begin{aligned} M_k(T;\alpha) &\gg_{q,k} \sum_{\chi} M_k(T;\chi) + o_{q,k}(T(\log T)^{k^2}) \\ &\geqslant M_k(T;\chi_0) + o_{q,k}(T(\log T)^{k^2}) \\ &\gg_{q,k} T(\log T)^{k^2}, \end{aligned}$$

completing the proof.

3.3 Proof of Theorem 3.1.1

The goal here is to compute the asymptotic for $M_2(T;\alpha)$, for $\alpha \in \mathbb{Q}$ originally proved (unpublished) in Andersson's thesis [1, pp. 71-72]. We reprove this here as, in the process, we will be able to verify that our conjectures for the constants $c_{\boldsymbol{\ell}}(q)$ and $c_k(\alpha)$ are correct when $|\boldsymbol{\ell}| \leqslant 2$ or $k \leqslant 2$. Further, our discussion will imply that the conclusion in Proposition 3.2.1 is true unconditionally if $k = 2$.

We note here that the previous two propositions show that the result of Proposition 3.2.1 can be obtained unconditionally when $k = 2$, which we had not shown previously. This is because the hypotheses $\mathsf{GRH}(q)$ or $\mathsf{Sp}(q,k)$ were used only in the proof of Lemma 2.2.2 and this use can be replaced by the above propositions, which trivially give the bound

$$\int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}\left(\tfrac{1}{2}+it\right)\right|^2 dt \ll_q T(\log T)^{\lambda}$$

for $\boldsymbol{\ell}$ satisfying $|\boldsymbol{\ell}| = 2$.

We now return to the proof of Theorem 3.1.1. We will set $k = 2$ in (3.4), and use the same classification for the different terms that arise in the right hand side of (3.4) as from §3.2.

We state some lemmata. Their proofs are analogous to the corresponding ones from §3.2 and hence the details are omitted.

Lemma 3.3.1. *Suppose that χ and ν are distinct characters modulo q. Then, for $s = 1/2 + it$,*

$$\int_T^{2T} L(s,\chi)^2\overline{L(s,\nu)}^2 \ll_q T(\log T)^2.$$

Proof. This is analogous to Lemma 3.2.4. □

Lemma 3.3.2. *Suppose that $\boldsymbol{\ell}^{(1)}$ and $\boldsymbol{\ell}^{(2)}$ are tuples of nonnegative integers indexed by characters modulo q satisfying $|\boldsymbol{\ell}^{(1)}| = |\boldsymbol{\ell}^{(1)}| = 2$. Further, suppose that $\boldsymbol{\ell}^{(1)}$ and $\boldsymbol{\ell}^{(2)}$ corresponds to a minor off-diagonal term. Then, for $s = 1/2 + it$,*

$$\int_T^{2T} \mathcal{L}^{\boldsymbol{\ell}^{(1)}}(s)\overline{\mathcal{L}^{\boldsymbol{\ell}^{(2)}}(s)}\, dt \ll_q T(\log T)^3.$$

Proof. This is analogous to Lemma 3.2.3. □

We can now prove the theorem. Putting $k = 2$ in (3.4), we get

$$\frac{q^2}{\varphi(q)^4} \sum_{\substack{|\boldsymbol{\ell}^{(1)}|=2,\\ |\boldsymbol{\ell}^{(2)}|=2}} \binom{k}{\boldsymbol{\ell}^{(1)}}\binom{k}{\boldsymbol{\ell}^{(2)}} \left[\prod_{\chi} \chi(a)^{\ell_\chi^{(2)}-\ell_\chi^{(1)}}\right] \int_T^{2T} \mathcal{L}^{\boldsymbol{\ell}^{(1)}}(s)\overline{\mathcal{L}^{\boldsymbol{\ell}^{(2)}}(s)}\, dt. \qquad (3.5)$$

We now use Proposition 2.2.3 to deal with the terms with the primary diagonal terms (i.e. those corresponding to $\boldsymbol{\ell}^{(1)} = \boldsymbol{\ell}^{(2)} = 2\boldsymbol{\delta}^{\chi}$). The discussion from §3.2 tells us that summing the main terms from Proposition 2.2.3 over χ contributes $c_2(\alpha)T(\log T)^4$, which gives the main term in Theorem 3.1.1. Note that this matches up with the conjectural constant from Theorem 3.1.2 for $k = 2$.

It remains to show that all the remaining terms can be absorbed in the error term in Theorem 3.1.1. We do this by applying Proposition 2.2.4, Lemma 3.3.1 and Lemma 3.3.2 appropriately to terms in (3.5), depending on their classification. There are $\ll \varphi(q)^4$ such terms in (3.5), and they each contribute at most $\ll_q T(\log T)^3$. This completes the proof.

4 Moments of the Hurwitz zeta function with irrational shifts and associated Diophantine equations

4.1 Introduction

The content of this chapter, where unremarked is joint work with Winston Heap and where specifically marked is joint work with Winston Heap and Trevor Wooley. Our objective in this chapter is to persuade the reader of the plausibility of Conjecture 1.0.2. The conjecture (joint with Winston Heap) states that

$$\frac{M_k(T;\alpha)}{T} = \frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it,\alpha)|^{2k}\,dt \sim k!(\log T)^k, \tag{4.1}$$

for any algebraic irrational α with degree $d \geqslant k$ and almost all transcendental α. This conjectural asymptotic seems surprising when compared to Conjecture 1.0.1, and the behaviour of moments in more arithmetic settings – it suggests that, on average, the Hurwitz zeta function with an irrational shift parameter is much smaller than with a rational shift. In fact, the probabilistically inclined reader may realize that the right hand side of (4.1) are the moments of a complex normal variable with mean 0 and variance $\log T$ unlike the case of $\zeta(s)$, which is log-normal (as per Selberg's central limit theorem, see [68] and references therein). Since the normal distribution is characterized by its moments, Conjecture 1.0.2 is tantamount to claiming that $t \mapsto \zeta(\frac{1}{2}+it,\alpha)$ is approximately normal[1] on $t \in [T,2T]$ when T is large.

The first piece of evidence for our conjecture is another conjecture: in his PhD thesis [1], Andersson conjectured that if one takes an average over α as well, then the the asymptotic holds. Concretely, he conjectured

$$\frac{1}{T}\int_1^2\int_T^{2T} |\zeta(\tfrac{1}{2}+it,\alpha)|^{2k}\,dt\,d\alpha \sim k!(\log T)^k.$$

He also proved this conjecture for $k=2$, while the case $k=1$ follows from Rane's asymptotic (3.1). Our conjecture can be viewed as a point-wise version of this: we expect the asymptotic to continue holding point-wise for a full measure set of α.

For our second piece of evidence we first introduce the pseudomoments of $\zeta(s,\alpha)$, $P_k(N;\alpha)$ defined by

$$P_k(N;\alpha) = \lim_{T\to\infty}\frac{1}{T}\int_T^{2T}\left|\sum_{0\leqslant n\leqslant N}\frac{1}{(n+\alpha)^{1/2+it}}\right|^{2k} dt. \tag{4.2}$$

In other words, the pseudomoment is obtained by taking a truncation to N of the generalized Dirichlet series of $\zeta(s,\alpha)$, computing the normalized moments at scale T, and then taking $T\to\infty$. This is similar to the analogous concept of pseudomoments $P_k^m(N)$ of $\zeta(s)^m$, where one replaces $\zeta(s)^m$ by a truncation of its Dirichlet series to get

$$P_k^m(N) = \lim_{T\to\infty}\frac{1}{T}\int_T^{2T}\left|\sum_{1\leqslant n\leqslant N}\frac{d_m(n)}{n^{1/2+it}}\right|^{2k} dt. \tag{4.3}$$

Evidently, $P_k^1(N) = P_k(N;1)$. Pseudomoments were first considered by Conrey and Gamburd [19] where they introduced $P_k^1(N)$ and proved that

$$P_k^1(N) \sim a(k)\gamma(k)(\log N)^{k^2},$$

[1] more precisely the complex normal $\mathcal{CN}(0,\log T)$ with independent real and imaginary parts

where $a(k)$ is the usual arithmetic factor (1.7) in the conjectural moments of $\zeta(s)$, and $\gamma(k)$ is the volume of a convex polytope. The reader should compare this to Conjecture 1.2.2. Pseudomoments of $\zeta(s)^m$ have since been studied by many authors [11, 10, 37, 13, 29, 30]. One reason for interest in pseudomoments is the connection to random multiplicative functions. Let $f : \mathbb{N} \to S^1$ be a Steinhaus random multiplicative function. That is,

$$\{f(p) : p \text{ prime}\},$$

is a sequence of prime-indexed i.i.d. Steinhaus random variables[2], and $f(n)$ for $n \in \mathbb{N}$ is defined by complete multiplicativity

$$f(n) = \prod_{p^\alpha \| n} f(p)^\alpha.$$

With these definitions, one has

$$P_k^m(N) = \mathbb{E}\Big[\Big|\sum_{n\leqslant N} \frac{d_m(n)f(n)}{n^{1/2}}\Big|^{2k}\Big]. \tag{4.4}$$

This relation is easy to see for integer k, as it follows from the orthogonality relations

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} \Big(\frac{n_1 n_2 \cdots n_k}{m_1 m_2 \cdots m_k}\Big)^{-it} dt = \mathbb{1}_{n_1 n_2 \cdots n_k = m_1 m_2 \cdots m_k},$$

and

$$\mathbb{E}\Big[\Big(f(n_1)f(n_2)\cdots f(n_k)\overline{f(m_1)f(m_2)\cdots f(m_k)}\Big)\Big] = \mathbb{1}_{n_1 n_2 \cdots n_k = m_1 m_2 \cdots m_k}$$

simply by expanding the squares in (4.3) and (4.4).

This connection to random multipicative functions is sometimes called the Bohr correspondence and is crucial in the Bohr–Jessen theory about the value distribution of $\zeta(s)$ (see, for example, [57, Chapter 3]).

[2] i.e., independent and uniformly distributed on the complex unit circle S^1

Returning now to (4.2), opening the squares, pushing the limit and integral inside, and using the orthogonality relation

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T}(x^{-it})dt = \mathbb{1}_{x=1},$$

we obtain

$$P_k(N;\alpha) = \sum_{\substack{0\leqslant m_j,n_j\leqslant N\\ 1\leqslant j\leqslant k}} \frac{\mathbb{1}_{(n_1+\alpha)\cdots(n_k+\alpha)=(m_1+\alpha)\cdots(m_k+\alpha)}}{(n_1+\alpha)^{1/2}\cdots(n_k+\alpha)^{1/2}(m_1+\alpha)^{1/2}\cdots(m_k+\alpha)^{1/2}}. \tag{4.5}$$

Now, if we assume that α is transcendental[3], then we find that there are no solutions to the equation

$$(n_1+\alpha)\cdots(n_k+\alpha) = (m_1+\alpha)\cdots(m_k+\alpha),$$

apart from the "diagonal" ones in which the tuple $(m_1,\dots,m_k)$ is a permutation of $(n_1,\dots,n_k)$, as any such solution would exhibit that α is a root of the non-zero polynomial

$$F(t) = F_{\mathbf{n},\mathbf{m}}(t) := \prod_{j=1}^k (t+n_j) - \prod_{j=1}^k (t+m_j),$$

contradicting the fact that α is transcendental[4].

For most $(n_1,\dots,n_k)$, there are $k!$ such permutations, from which it follows from (4.5) that

$$P_k(N;\alpha) \sim k! \sum_{\substack{0\leqslant n_j\leqslant N\\ 1\leqslant j\leqslant k}} \frac{1}{\prod_j (n_j+\alpha)} = k!\Big(\sum_{0\leqslant n\leqslant N}\frac{1}{n+\alpha}\Big)^k,$$

and hence we find that $P_k(N;\alpha) \sim k!(\log N)^k$. This is our second piece of evidence for Conjecture 1.0.2. In §4.2, we provide another interpretation of this calculation that explains the occurrence of Gaussian moments in a more natural way.

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[3]Assuming that the degree of α, d satisfies $d \geqslant k$ is enough, interpreting the degree of a transcendental number as being $d=\infty$.

[4]If we assumed that α is algebraic with $d\geqslant k$ then the contradiction relies on the fact that the degree of $F(t)$ is strictly smaller than k.

This previous calculation leads naturally to the question of finding solutions to the Diophantine equation

$$(x_1+\alpha)\cdots(x_k+\alpha)=(y_1+\alpha)\cdots(y_k+\alpha), \tag{4.6}$$

with $1 \leqslant x_j, y_k \leqslant N$ with $1 \leqslant j \leqslant k$; this will lead to our third piece of evidence. Note that we previously allowed for some variables to be 0 but we have removed this case from the analysis for convenience. Our results will be valid for all irrational $\alpha \in \mathbb{C}$, so one may recover this case by replacing (α, N) with $(\alpha-1, N+1)$. This Diophantine equation was considered in joint work [42] of the author with Winston Heap and Trevor Wooley[5]. As discussed in Chapter 1, this problem was also considered independently by Bourgain, Garaev, Konyagin, and Shparlinski [12], and our proofs are essentially the same.

The problem of finding integral solutions to (4.6) is complicated by the fact that if the degree d of α satisfies $d<k$ then the degree of $F_{\mathbf{x},\mathbf{y}}(t)$ above is high enough for it to be potentially divisible by the minimal polynomial of α, which means it is possible to have non-trivial solutions to (4.6), i.e., solutions in which $(x_1,\ldots,x_k)$ is *not* a permutation of $(y_1,\ldots,y_k)$. To state our results in this regard, we introduce, for $\nu \in \mathbb{Z}[\alpha]$,

$$\tau_k(\nu;N,\alpha)=\sum_{\substack{1\leqslant d_1\leqslant N \\ (d_1+\alpha)\cdots(d_k+\alpha)=\nu}}\cdots\sum_{1\leqslant d_k\leqslant N}1.$$

One may interpret this as a k-fold divisor function in $\mathbb{Z}[\alpha]$, where one has restricted to divisors in the set $\{n+\alpha : n\leqslant N, n\in\mathbb{N}\}$. We also introduce $T_k(N)$ to denote the number of pairs of k-tuples $\mathbf{x}=(x_1,\ldots,x_k)$ and $\mathbf{y}=(y_1,\ldots,y_k)$ such that $\mathbf{x}$ is a permutation of $\mathbf{y}$. Clearly, $T_k(N)=k!N^k+O_k(N^{k-1})$, with the main contribution coming from those $\mathbf{x}$ in which $x_1,x_2,\cdots,x_k$ are distinct.

[5]The notation here diverges from [42] where necessary to remain consistent with the rest of this thesis.

The mean value

$$\sum_{\nu \in \mathbb{Z}[\alpha]} \tau_k(\nu; N, \alpha)^2$$

now counts the number of integral solutions of (4.6) with $1 \leqslant x_j, y_j \leqslant N$ for $1 \leqslant j \leqslant k$. Our main result in this regard states that when $\alpha \notin \mathbb{Q}$, then almost all solutions of (4.6) are the diagonal ones in which $\mathbf{x}$ is a permutation of $\mathbf{y}$. Thus, almost all elements $\nu \in \mathbb{Z}[\alpha]$ that may be written as a k-fold product of shifted integers $n + \alpha$ are represented essentially uniquely in this manner.

The result may then be stated as follows.

Theorem 4.1.1 (Heap–S.–Wooley, 2023)**.** *Let $k \in \mathbb{N}$ and $\epsilon > 0$. Suppose that $\alpha \in \mathbb{C}$ is algebraic of degree d over $\mathbb{Q}$, where $2 \leqslant d < k$. Then one has*

$$\sum_{\nu \in \mathbb{Z}[\alpha]} \tau_k(\nu; N, \alpha)^2 = T_k(N) + O_{k,\alpha,\epsilon}(N^{k-d+1+\epsilon}).$$

If instead $\alpha \in \mathbb{C}$ is either transcendental, or else algebraic of degree d over $\mathbb{Q}$ with $d \geqslant k$, then one may omit the error term.

It follows that when $\alpha \notin \mathbb{Q}$, then there is a paucity of non-diagonal solutions in the equation (4.6). Moreover, one has the asymptotic formula

$$\sum_{\nu \in \mathbb{Z}[\alpha]} \tau_k(\nu; X, \alpha)^2 = k! X^k + O_{k,\epsilon}(X^{k-1+\epsilon})$$

for $\alpha \notin \mathbb{Q}$. These conclusions are in marked contrast with the corresponding situation in which $\alpha \in \mathbb{Q}$. When α is rational, experts will recognise that a straightforward exercise employing the circle method yields the lower bound

$$\sum_{\nu \in \mathbb{Z}[\alpha]} \tau_k(\nu; X, \alpha)^2 \gg_{\alpha,k} X^k (\log X)^{(k-1)^2}.$$

Indeed, additional work would exhibit an asymptotic formula in place of this lower bound. In this regard, we note that the contour integral methods of [35, 44] would also be accessible. The inquisitive reader interested in paucity problems for diagonal Diophantine systems will find a representative slice of the relevant literature in [76, 77, 83, 86].

The basic strategy that we employ in the proof of Theorem 4.1.1 is inspired by an examination of the polynomial $F_{\mathbf{x},\mathbf{y}}(t)$. There are parallels here with the treatment of Vaughan and Wooley of the Vinogradov system [83]. The full proof is presented in §4.3.

This completes our presentation of the evidence of Conjecture 1.0.2. In ongoing work with Winston Heap [41], we are considering the case $k = 2$ of this conjecture.

4.2 Pseudomoments via the Central Limit Theorem

To provide a natural reason for the appearance of Gaussian behaviour in the moments, we need the following consequence of Lyapunov's criterion for the central limit theorem [9, Theorem 27.3].

Lemma 4.2.1. *Let $\{a_n\}_{n\geqslant 0}$, be a sequence of real numbers satisfying*

$$\sum_{n\geqslant 0} |a_n|^2 = \infty,$$

and for some $\delta > 0$,

$$\sum_{n\geqslant 0} |a_n|^{2+\delta} < \infty.$$

Further, suppose that $\{X_n\}_{n\geqslant 0}$ is a sequence of i.i.d. real-valued (resp. circularly-symmetric[6] complex-valued) random variables that has mean 0 and variance 1. Then, if

$$S_N = \sum_{0\leqslant n\leqslant N} a_n X_n,$$

[6]A complex random variable Z is circularly-symmetric if $e^{i\theta}Z$ and Z have the same distribution for any fixed $\theta \in [0, 2\pi)$.

and

$$\widehat{S_N} = \frac{S_N - \mathbb{E}[S_N]}{\sqrt{\mathbb{V}[S_N]}},$$

then $\widehat{S_N}$ *converges in distribution to* $\mathcal{N}(0,1)$ *(resp.* $\mathcal{CN}(0,1)$*) as* $N \to \infty$*.*

Proof. First we consider the case of real-valued random variables. Since the X_n are independent and $\mathbb{E}[X_n] = 0$, it follows that

$$\mathbb{V}[S_N] = \sum_{0 \leqslant n \leqslant N} |a_n|^2,$$

while, by the monotonicity of L^p norms in probability spaces,

$$\sum_{0 \leqslant n \leqslant N} \mathbb{E}[|a_n X_n - \mathbb{E}(a_n X_n)|^{2+\delta}] \leqslant \sum_{0 \leqslant n \leqslant N} |a_n|^{2+\delta}.$$

From the hypotheses on $\{a_j\}_{j \geqslant 0}$, it is now immediate that

$$\lim_{n \to \infty} \frac{1}{\mathbb{V}[S_N]^{2+\delta}} \sum_{0 \leqslant n \leqslant N} \mathbb{E}[|a_n X_n - \mathbb{E}(a_n X_n)|^{2+\delta}] = 0,$$

verifying Lyapunov's criterion for the central limit theorem (see, for example, [9, Theorem 27.3]). From this the real case of the lemma follows.

The circularly-symmetric complex case follows by applying the real case to the real and imaginary parts of X_n. The variance of $\Re(X_n), \Im(X_n)$ will now be $1/2$ instead of 1. This is because, $\mathbb{V}[\Re(X_n)] = \mathbb{V}[\Im(X_n)]$ and $\mathbb{E}[\Re(X_n)] = \mathbb{E}[\Im(X_n)]$ due to circular-symmetry, and hence

$$1 = \mathbb{V}[X_n] = \mathbb{E}[|\Re(X_n)|^2 + |\Im(X_n)|^2] = \mathbb{V}[\Re(X_n)] + \mathbb{V}[\Im(X_n)] = 2\mathbb{V}[\Re(X_n)].$$

One may rescale the random variable by incorporating this factor into the coefficients a_n. This implies that X_n converges to a complex normal. The independence of the real and imaginary parts of this normal follows from the circular-symmetry of X_n, as the limit itself must be circularly-symmetric. This completes the proof. $\square$

Now, by recalling the ideas behind the Bohr correspondence, one can see that if α is transcendental, then the set in (1.2) is linearly independent over $\mathbb{Q}$, and hence the Kronecker–Weyl theorem tells us that as $T \to \infty$,

$$\{t \in [T, 2T] \mapsto (n+\alpha)^{-it}\}_{0 \leqslant n \leqslant N} \xrightarrow{\mathrm{d}} \{X_n\}_{0 \leqslant n \leqslant N},$$

where the X_n are i.i.d. random variables taking values uniformly on S^1, and hence are circularly-symmetric with mean 0 and variance 1.

From this limit in distribution, we find that

$$P_k(N; \alpha) = \mathbb{E}\Big[\Big| \sum_{0 \leqslant n \leqslant N} \frac{X_n}{(n+\alpha)^{1/2}} \Big|^{2k}\Big].$$

However, an application of Lemma 4.2.1 with $a_n = (n+\alpha)^{-1/2}$ and any $\delta > 0$ tells us that

$$\sum_{0 \leqslant n \leqslant N} \frac{X_n}{(n+\alpha)^{1/2}},$$

is essentially normal with mean 0 and variance $\log N$ as $N \to \infty$, from which we conclude that

$$P_k(N; \alpha) \sim k!(\log N)^k,$$

as desired.

4.3 Proof of Theorem 4.1.1

The case in which $\alpha \in \mathbb{C}$ is either transcendental, or else is algebraic of degree $d \geqslant k$ over $\mathbb{Q}$ follows from the argument presented in the introduction to this chapter. We present the argument here slightly differently to set the stage for the case in which $d < k$.

We rewrite the equation (4.6) by using elementary symmetric polynomials $\sigma_j(\mathbf{z}) \in \mathbb{Z}[z_1, \ldots, z_k]$. These may be defined for $j \geqslant 0$ by means of the generating function identity

$$\sum_{j=0}^{k} \sigma_j(\mathbf{z}) t^{k-j} = \prod_{i=1}^{k} (t + z_i).$$

The equation (4.6) may thus be rewritten in the form

$$\sum_{j=0}^{k} \sigma_j(\mathbf{x}) \alpha^{k-j} = \sum_{j=0}^{k} \sigma_j(\mathbf{y}) \alpha^{k-j}.$$

Since $\sigma_0(\mathbf{x}) = 1 = \sigma_0(\mathbf{y})$, we find that

$$\sum_{j=1}^{k} (\sigma_j(\mathbf{x}) - \sigma_j(\mathbf{y})) \alpha^{k-j} = 0. \tag{4.7}$$

In our present situation with α either transcendental, or else algebraic of degree $d \geqslant k$ over $\mathbb{Q}$, the complex numbers $1, \alpha, \ldots, \alpha^{k-1}$ are linearly independent over $\mathbb{Q}$. Then it follows from (4.7) that $\sigma_j(\mathbf{x}) = \sigma_j(\mathbf{y})$ $(1 \leqslant j \leqslant k)$. In particular, over the ring $\mathbb{Z}[t]$ one obtains the polynomial identity

$$\prod_{j=1}^{k} (t - x_j) = \prod_{j=1}^{k} (t - y_j). \tag{4.8}$$

The polynomial relation (4.8) implies that left and right hand sides must have the same zeros with identical multiplicities. Hence $(x_1, \ldots, x_k)$ must be a permutation of $(y_1, \ldots, y_k)$. The conclusion

$$\sum_{\nu \in \mathbb{Z}[\alpha]} \tau_k(\nu; N, \alpha)^2 = T_k(N)$$

is then immediate on recalling the Diophantine interpretation (4.6) of the mean value on the left hand side.

We now assume that $\alpha \in \mathbb{C}$ is an algebraic number of degree d over $\mathbb{Q}$, with $2 \leqslant d < k$. In this situation the equation (4.6) simplifies, since α^d may be expressed as a $\mathbb{Q}$-linear combination of $1, \alpha, \ldots, \alpha^{d-1}$. However, the equation

(4.7) no longer delivers k independent polynomial equations, but instead d such equations with $d < k$. The previous strategy is thus no longer applicable.

Let $\mathbf{x}, \mathbf{y}$ be an integral solution of the equation (4.6) with $1 \leqslant x_i, y_i \leqslant X$ $(1 \leqslant i \leqslant k)$, in which $(x_1, \ldots, x_k)$ is not a permutation of $(y_1, \ldots, y_k)$. Observe first that if $x_i = y_j$ for any indices i and j with $1 \leqslant i, j \leqslant k$, then we may cancel the factors $x_i + \alpha$ and $y_j + \alpha$, respectively, from the left and right hand sides of (4.6). It thus suffices to establish the conclusion of Theorem 4.1.1 with k replaced by $k-1$. Here, of course, if $d \geqslant k-1$, then the desired conclusion follows from the previous discussion. By repeatedly cancelling pairs of equal factors in this way, it is apparent that there is no loss of generality in supposing henceforth that $x_i = y_j$ for no indices i and j with $1 \leqslant i, j \leqslant k$.

Consider the polynomial

$$F(t) = \prod_{i=1}^{k}(t + x_i) - \prod_{i=1}^{k}(t + y_i). \tag{4.9}$$

This polynomial has degree at most $k-1$, and so for suitable integers $a_j = a_j(\mathbf{x}, \mathbf{y})$ $(0 \leqslant j \leqslant k-1)$, we may write

$$F(t) = a_0 + a_1 t + \ldots + a_{k-1} t^{k-1}.$$

Note that for $0 \leqslant j \leqslant k-1$, one has

$$|a_j| = |\sigma_{k-j}(\mathbf{x}) - \sigma_{k-j}(\mathbf{y})| \ll N^{k-j}. \tag{4.10}$$

Next, denote by $m_\alpha \in \mathbb{Z}[t]$ the minimal polynomial of α over $\mathbb{Z}$. Then m_α is irreducible of degree d over $\mathbb{Z}$, and if m_α has leading coefficient $c_d \neq 0$, then $c_d^{-1} m_\alpha \in \mathbb{Q}[t]$ is the usual minimal polynomial of α over $\mathbb{Q}$. We may write

$$m_\alpha(t) = c_0 + c_1 t + \ldots + c_d t^d,$$

in which $|c_j| \ll_\alpha 1$ $(0 \leqslant j \leqslant d)$. We observe from (4.6) and (4.9) that

$$F(\alpha) = \prod_{i=1}^{k}(x_i + \alpha) - \prod_{i=1}^{k}(y_i + \alpha) = 0,$$

whence $m_\alpha(t)$ divides $F(t)$. Consequently, there is a polynomial

$$\Psi(t) = \Psi_\alpha(t; \mathbf{x}, \mathbf{y}) \in \mathbb{Z}[t]$$

having the property that

$$F(t) = m_\alpha(t)\Psi(t). \tag{4.11}$$

Since $\deg(\Psi) = \deg(F) - \deg(m_\alpha) \leqslant k - 1 - d$, we may write

$$\Psi(t) = b_0 + b_1 t + \ldots + b_{k-1-d} t^{k-1-d},$$

where $b_m \in \mathbb{Z}$ $(0 \leqslant m \leqslant k-1-d)$. Our immediate goal is to bound the coefficients b_m.

We claim that for $0 \leqslant m \leqslant k - d - 1$, one has

$$|b_m| \ll N^{k-d-m}. \tag{4.12}$$

This we establish by considering the formal Laurent series for $m_\alpha(t)^{-1}$. Thus, we have $m_\alpha(t)^{-1} = e(t) \in \mathbb{Q}((1/t))$, where for suitable rational coefficients $e_j \in \mathbb{Q}$ $(j \geqslant d)$ one has

$$e(t) = \sum_{j=d}^{\infty} e_j t^{-j} = \frac{1}{c_d t^d}(1 + c_d^{-1} c_{d-1} t^{-1} + \ldots + c_d^{-1} c_0 t^{-d})^{-1}.$$

Note here that $c_d \neq 0$. Further, since $c_0, \ldots, c_d$ depend at most on α, it follows from a Taylor series expansion that $e_j \ll_{\alpha,j} 1$. We may therefore infer from (4.11) that $\Psi(t) = e(t)F(t)$, whence

$$\sum_{m=0}^{k-1-d} b_m t^m = \left(\sum_{j=d}^{\infty} e_j t^{-j} \right) \left(\sum_{i=0}^{k-1} a_i t^i \right).$$

In view of the bounds (4.10) and $e_j \ll_{\alpha,j} 1$, we deduce that for $0 \leqslant m \leqslant k - 1 - d$ one has

$$\begin{aligned} b_m &= e_d a_{m+d} + e_{d+1} a_{m+d+1} + \ldots + e_{k-1-m} a_{k-1} \\ &\ll N^{k-d-m} + N^{k-d-m-1} + \ldots + N \ll N^{k-d-m}. \end{aligned}$$

This confirms the bound (4.12). We may suppose henceforth that there is a positive number $C = C(k, \alpha)$ having the property that

$$|b_m| \leqslant CN^{k-d-m} \quad (0 \leqslant m \leqslant k-d-1). \tag{4.13}$$

We now arrive at the polynomial identity that does the heavy lifting in the proof of Theorem 4.1.1.

Lemma 4.3.1. *Suppose that* $\mathbf{x}, \mathbf{y}$ *is an integral solution of the equation* (4.6) *with* $1 \leqslant x_i, y_i \leqslant N$ $(1 \leqslant i \leqslant k)$*, in which* $x_i = y_j$ *for no indices* i *and* j *with* $1 \leqslant i, j \leqslant k$*. Then, for each index* j *with* $1 \leqslant j \leqslant k$*, there is an integer* ρ_j*, with* $1 \leqslant |\rho_j| \leqslant kCN^{k-d}$*, having the property that*

$$\prod_{i=1}^{k} (x_i - y_j) = \rho_j m_\alpha(-y_j).$$

Proof. Recalling (4.9) and (4.11), we see that

$$F(-y_j) = \prod_{i=1}^{k} (x_i - y_j) = m_\alpha(-y_j)\Psi(-y_j).$$

But in view of (4.13), one has

$$|\Psi(-y_j)| \leqslant \sum_{m=0}^{k-1-d} |b_m| y_j^m \leqslant (k-d)CN^{k-d}.$$

Thus, there is an integer $\rho_j = \Psi(-y_j)$ with $|\rho_j| \leqslant kCN^{k-d}$ for which

$$\prod_{i=1}^{k} (x_i - y_j) = m_\alpha(-y_j)\rho_j.$$

Notice here that since the left hand side is a non-zero integer, then so too are both factors on the right hand side. The conclusion of the lemma follows. □

We may now complete the proof of Theorem 4.1.1. Our previous discussion ensures that it is sufficient to count solutions $\mathbf{x}, \mathbf{y}$ of (4.6) with $1 \leqslant x_i, y_i \leqslant N$ $(1 \leqslant i \leqslant k)$, in which $x_i = y_j$ for no indices i and j with $1 \leqslant i, j \leqslant k$. Given any such solution, an application of Lemma 4.3.1 with $j = k$ shows that, for some integer ρ_k with $1 \leqslant |\rho_k| \leqslant kCN^{k-d}$, one has

$$\prod_{i=1}^{k}(x_i - y_k) = \rho_k m_\alpha(-y_k). \tag{4.14}$$

Fix any one of the $O(N)$ possible choices for y_k, and likewise any one of the $O(N^{k-d})$ possible choices for ρ_k. Then we see from (4.14) that each of the factors $x_i - y_k$ $(1 \leqslant i \leqslant k)$ must be a divisor of the non-zero integer $M = \rho_k m_\alpha(-y_k)$. It therefore follows from an elementary estimate for the divisor function that there are $O(M^\epsilon)$ possible choices for $x_i - y_k$ $(1 \leqslant i \leqslant k)$. Fix any one such choice. Then since y_k has already been fixed, we see that $x_1, \ldots, x_k$ and y_k are now all fixed.

At this point we return to the equation (4.6). By taking norms from $\mathbb{Q}(\alpha)$ down to $\mathbb{Q}$, we see that

$$\prod_{i=1}^{k} m_\alpha(-y_i) = \prod_{i=1}^{k} m_\alpha(-x_i).$$

The right hand side here is already fixed and non-zero. A divisor function estimate therefore shows that there are $O(N^\epsilon)$ possible choices for integers $n_1, \ldots, n_k$ having the property that

$$m_\alpha(-y_i) = n_i \quad (1 \leqslant i \leqslant k).$$

Fixing any one such choice for the k-tuple $\mathbf{n}$, we find that when $1 \leqslant i \leqslant k$, there are at most d choices for the integer solution y_i of the polynomial equation $m_\alpha(-t) = n_i$. Altogether then, the number of possible choices for $\mathbf{x}$ and $\mathbf{y}$ given a fixed choice for y_k and ρ_k is $O((MN)^\epsilon)$. Thus we conclude that the total number of possible choices for $\mathbf{x}$ and $\mathbf{y}$ is $O(N^{k-d+1+\epsilon})$, and hence

$$\sum_{\nu \in \mathbb{Z}[\alpha]} \tau_k(\nu; N, \alpha)^2 - T_k(N) \ll N^{k-d+1+\epsilon}.$$

This completes the proof of Theorem 4.1.1.

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