MA 162 Exam 1 Solutions
June 25, 2019

Problem 1. Using the vectors below, determine which statement is true

$$\mathbf{a} = \langle 1, -1, 1\rangle, \quad \mathbf{b} = \langle 3, -1, -2\rangle, \quad \mathbf{c} = \langle 3, 5, 2\rangle, \quad \mathbf{d} = \langle 7, -1, -8\rangle.$$

Solution. Take all possible dot products:

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \langle 1, -1, 1\rangle \cdot \langle 3, -1, -2\rangle = 3 + 1 - 2 = 2,\\
\mathbf{a}\cdot\mathbf{c} &= \langle 1, -1, 1\rangle \cdot \langle 3, 5, 2\rangle = 3 - 5 + 2 = 0,\\
\mathbf{a}\cdot\mathbf{d} &= \langle 1, -1, 1\rangle \cdot \langle 7, -1, -8\rangle = 7 + 1 - 8 = 0,\\
\mathbf{b}\cdot\mathbf{c} &= \langle 3, -1, -2\rangle \cdot \langle 3, 5, 2\rangle = 9 - 5 - 4 = 0,\\
\mathbf{b}\cdot\mathbf{d} &= \langle 3, -1, -2\rangle \cdot \langle 7, -1, -8\rangle = 27 + 1 + 16 = 44.
\end{aligned}$$

From our calculations above we see that $\mathbf{a}$ is orthogonal to and $\mathbf{d}$, and $\mathbf{b}$ is orthogonal to $\mathbf{c}$. Therefore, the only true statement is (B) $\mathbf{a}$ *is orthogonal to* $\mathbf{c}$ *and* $\mathbf{d}$. ◇

Problem 2. The curve $y = \ln(\cos x)$, $0 \le x \le \pi/4$, is rotated about the x-axis. The surface area of the resulting solid could be calculated by

Solution. Recall that the surface area for a surface of revolution (about the x-axis) is given by the formula

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

First we find he derivative of f, which is

$$f'(x) = -\frac{\sin(x)}{\cos x} = -\tan x.$$

Next we plug f and f' into the formula

$$\begin{aligned}
S &= \int_0^{\pi/4} 2\pi \ln(\cos x)\sqrt{1 + (-\tan x)^2}\,dx\\
&= \int_0^{\pi/4} 2\pi \ln(\cos x)\sqrt{1 + \tan^2 x}\,dx\\
&= \int_0^{\pi/4} 2\pi \ln(\cos x)\sqrt{\sec^2 x}\,dx,
\end{aligned}$$

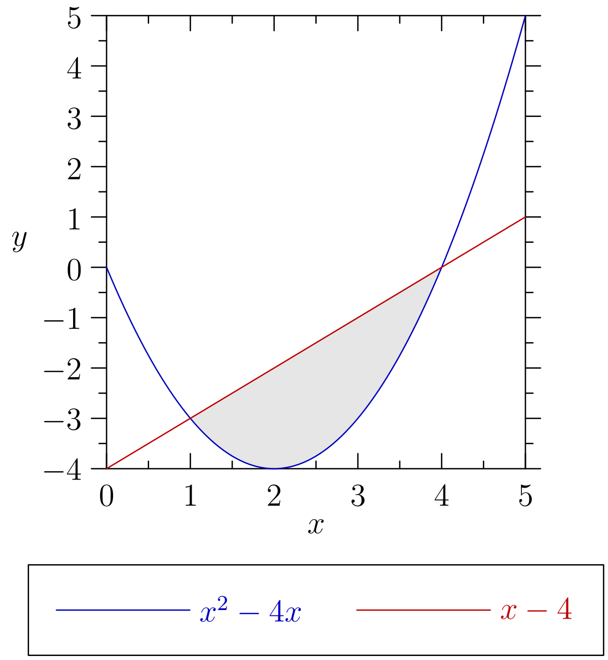
and use the identity $1 + \tan^2 x = \sec^2 x$, to arrive at the answer

$$= \int_0^{\pi/4} 2\pi \ln(\cos x) \sec x \, dx.$$

◇

Problem 3. Find the area of the region between the curves $y = x^2 - 4x$ and $y = x - 4$.

Solution. First we sketch the region in question, it is shaded gray in the figure below



From that same image, we see that the area under the curve is given by

$$\int_1^4 x - 4 - (x^2 - 4x) \, dx.$$

Computing this integral, we get

$$\begin{aligned}
\int_1^4 x - 4 - (x^2 - 4x)\,dx &= \int_1^4 x - 4 - x^2 + 4x\,dx \\
&= \int_1^4 -x^2 + 5x - 4\,dx \\
&= -\tfrac{1}{3}x^3 + \tfrac{5}{2}x^2 - 4x\Big|_1^4 \\
&= -\tfrac{1}{3}4^3 + \tfrac{5}{2}4^2 - 4\cdot 4 - \left(-\tfrac{1}{3}1^4 + \tfrac{5}{2}\cdot 1^2 - 4\cdot 1\right) \\
&= -\tfrac{64}{3} + 40 - 16 + \tfrac{1}{3} - \tfrac{5}{2} + 4 \\
&= -21 + 40 - 16 + 4 - \frac{5}{2} \\
&= \frac{14}{2} - \frac{5}{2} \\
&= \frac{9}{2}.
\end{aligned}$$

◇

Problem 4. Evaluate

$$\int_0^{\pi/2} x\cos x\,dx.$$

Solution. We evaluate the integral by the method of tabular integration. First, we fill in the table

+	x	$\cos x$
-	1	$\sin x$
+	0	$-\cos x$

.

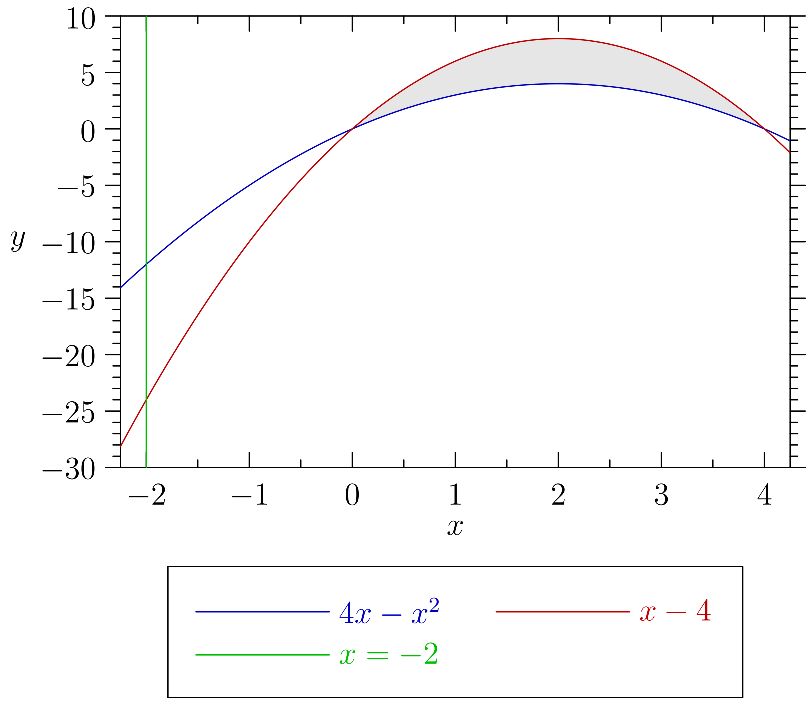
Then we use it to show that

$$\int_0^{\pi/2} x\cos x\,dx = x\sin x + \cos x\Big|_0^{\pi/2} = \tfrac{\pi}{2} - 0 - (0+1) = \tfrac{\pi}{2} - 1.$$

◇

Problem 5. Using the method of cylindrical shells, how could you find the volume of the region generated by rotating the region bounded by $y = 4x - x^2$ and $y = 8x - 2x^2$ about the line $x = -2$.

Solution. First, we sketch the graph of the region. The region in question is shaded gray in the figure below



From this image, and by straightforward calculations, we can see that $x = 0$ and $x = 4$ will be the bounds of our integral. Moreover, from the image we can see that the radius of the cylinder will be

$$r(x) = x - (-2) = x + 2$$

and the height

$$h(x) = 4x - x^2 - (8x - 2x^2) = x^2 - 4x.$$

so the volume of the solid of revolution is

$$\begin{aligned} V &= \int_0^4 2\pi r(x)h(x)\,dx \\ &= \int_0^4 2\pi(x+2)(x^2-4x)\,dx \\ &= 2\pi\int_0^4 (4x-x^2)(x+2)\,dx. \end{aligned}$$

◇

Problem 6. Find the arclength of $y = \frac{1}{4}(e^{2x} + e^{-2x})$ for $0 \le x \le 1$.

Solution. By the arclength formula

$$A = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Therefore,

$$f'(x) = \tfrac{1}{2}(e^{2x} - e^{-2x})$$

and

$$\begin{aligned}
A &= \int_0^1 \sqrt{1 + \tfrac{1}{4}(e^{2x} - e^{-2x})^2}\, dx \\
&= \int_0^1 \sqrt{1 + \tfrac{1}{4}e^{4x} - \tfrac{1}{2} + \tfrac{1}{4}e^{-4x}}\, dx \\
&= \int_0^1 \sqrt{\tfrac{1}{4}(2 + e^{4x} + e^{-4x})}\, dx \\
&= \int_0^1 \tfrac{1}{2}\sqrt{(e^{2x} + e^{-2x})^2}\, dx \\
&= \int_0^1 \tfrac{1}{2}(e^{2x} + e^{-2x})\, dx \\
&= \tfrac{1}{2}\Big(e^{2x} + e^{-2x}\Big|_0^1\Big) \\
&= \tfrac{1}{2}\Big(e^2 + e^{-2} - (1+1)\Big) \\
&= \tfrac{1}{2}(e^2 + e^{-2} + 2) \\
&= \frac{e^2 + e^{-2}}{2} + 1.
\end{aligned}$$

$\diamond$

Problem 7. Evaluate

$$\int \tan^4 x\, dx.$$

Solution. By using reduction formulas we learned in class

$$\begin{aligned}
\int \tan^4 x\, dx &= \tfrac{1}{3}\tan^3 x - \int \tan^2 x\, dx \\
&= \tfrac{1}{3}\tan^3 x - \left(\tan x - \int dx\right) \\
&= \tfrac{1}{3}\tan^3 x - \tan x + x + C.
\end{aligned}$$

If you did not remember the formula, repeatedly applying identities would suffice: first make the substitution $\tan^2 x = \sec^2 x - 1$ so

$$\begin{aligned}\int \tan^4 x\,dx &= \int (\sec^2 x - 1)\tan^2 x\,dx \\ &= \underbrace{\int \tan^2 x \sec^2 x}_{I} - \underbrace{\int \tan^2 x\,dx}_{J}.\end{aligned}$$

The integral I can easily integrated by making the substitution $u = \tan x$, $du = \sec^2 x\,dx$, as follows

$$I = \int u^2\,du = \tfrac{1}{3}u^3 = \tfrac{1}{3}\tan^3 x + C_I.$$

For J, we again use the identity $\tan^2 x = \sec^2 x - 1$ to get

$$J = \int \sec^2 x - 1\,dx = \int \sec^2 x\,dx - \int dx = \tan x - x + C_J.$$

Thus, the full integral is

$$\int \tan^4 x\,dx = I - J = \tfrac{1}{3}\tan^3 x - \tan x + x + C.$$

◇

Problem 8. Evaluate

$$\int_1^2 3x^2 \ln(4x)\,dx$$

Solution. We solve for this integral by integration by parts. First, let us find the form of the improper integral: with $u = \ln(4x)$, $du = 1/x$, $dv = 3x^2$, $v = x^3$, we have

$$\begin{aligned}\int 3x^3 \ln(4x)\,dx &= x^3\ln(4x) - \int x^2\,dx \\ &= x^3\ln(4x) - \tfrac{1}{3}x^3 \\ &= x^3\big(\ln(4x) - \tfrac{1}{3}\big).\end{aligned}$$

Thus, the integral is

$$\int_1^2 3x^3 \ln(4x)\,dx = 2^3(\ln 8 - \tfrac{1}{3}) - 1^3(\ln 4 - \tfrac{1}{3}) = 15\ln(4) - \tfrac{7}{3}.$$

◇

Problem 9. A tank in the shape of an inverted circular cone, with a radius of 2 m and a height of 5 m, is filled with water. The water is to be pumped out through the top of the cone. Set up but do not evaluate an integral to calculate the amount of work done in pumping out all of the water. The density of water is 1000 kg/m^3 and $g = 9.8$ m/s^2.

Solution. We have done this same problem before, and in full generality. Refer to Lecture 4 notes, Eq. 7 for the details. The work done is

$$W = \int_0^5 9800\pi(\tfrac{4}{25})y^2(5-y)\,dy.$$

◇

Problem 10. A cable 10 meters long with density 2 kg/m is hanging from the top of the roof 10 m above the ground. How much work is done in raising the cable to the top of the roof? ($g = 9.8$ m/s^2).

Solution. We solved a problem just like this before the exam. The work done is given by

$$\begin{aligned} W &= \int_0^{10} 19.6(10-y)\,dy \\ &= 19.6\int_{10}^{0} -u\,du \\ &= 19.6\int_0^{10} u\,du \\ &= 9.8u^2\Big|_0^{10} \\ &= 9.8 \cdot 10^2 \\ &= 980. \end{aligned}$$

◇