MA 162 Exam 2 Solutions
July 9, 2019

Problem 1. Given $a_n = \dfrac{2n+3}{4-n}$, then

Solution. By l'Hôpital's rule,

$$\lim_n a_n = \lim_n \frac{2}{-1} = -2$$

so the sequence converges.

But because the sequence converges to a nonzero real number, the series $\sum_n a_n$ must diverge by the quick test for divergence. $\diamond$

Problem 2. Let $f(x)$ be a function defined for $x \geq 1$, such that $1/\sqrt{2} \leq f(x) \leq 1$ for all $x \geq 1$. What can be said about the series

$$S_1 = \sum_{k=1}^{\infty} \frac{f(k)}{\sqrt{k}}, \qquad S_2 = \sum_{k=1}^{\infty} \frac{f(k)}{k^2}?$$

Solution. By the squeeze theorem for series, for S_1

$$\sum_{k=1}^{\infty} \frac{1/\sqrt{k}}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k} \leq S_1 \leq \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}},$$

so S_1 diverges since the series to the left of S_1 diverges (it is the harmonic series). For S_2,

$$\sum_{k=1}^{\infty} \frac{1/\sqrt{k}}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^{5/2}} \leq S_2 \leq \sum_{k=1}^{\infty} \frac{1}{k^2},$$

so S_2 converges since both of the series 'squeezing' S_2 converge by the p-series test. $\diamond$

Problem 3. Find the sum of the following series.

$$\sum_{k=1}^{\infty} \left(\frac{3 + 3^{2k+1}}{10^k} \right).$$

Solution. So far we only know how to deal with geometric series, i.e.

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad \text{for } |r| < 1,$$

so we need to reduce the series to a combination of geometric series. We do this as follows, first note that

$$\sum_{k=1}^{\infty}\left(\frac{3+3^{2k+1}}{10^k}\right)=\sum_{k=1}^{\infty}\frac{3}{10^k}+\sum_{k=1}^{\infty}\frac{3^{2k+1}}{10^k}$$

also by exponent laws, $3^{2k+1}=3\cdot 3^{2k}=3\cdot 9^k$, so

$$=\underbrace{3\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^k}_{S_1}+\underbrace{3\sum_{k=1}^{\infty}\left(\frac{9}{10}\right)^k}_{S_2}.$$

Let us deal with S_1 first. Note that $k=1$ so we need to step the series back. We do this as follows, first expand the series

$$\begin{aligned}S_1&=3\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots\right)\\&=\frac{3}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+\cdots\right)\\&=\frac{3}{10}\sum_{k=0}^{\infty}\left(\frac{1}{10}\right)^k\\&=\left(\frac{3}{10}\right)\left(\frac{1}{1-1/10}\right)\\&=\left(\frac{3}{10}\right)\left(\frac{10}{9}\right)\\&=\frac{1}{3}.\end{aligned}$$

Now we deal with S_2. As we saw for S_1, $k=1$ means that we need to take out a single factor of r from the series to step k back to 0. Therefore,

$$\begin{aligned}S_2&=\frac{27}{10}\sum_{k=0}^{\infty}\left(\frac{9}{10}\right)^k\\&=\left(\frac{27}{10}\right)\left(\frac{1}{1-9/10}\right)\\&=\left(\frac{27}{10}\right)(10)\\&=27.\end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty}\left(\frac{3+3^{2k+1}}{10^k}\right) = \frac{1}{3} + 27 = \frac{1}{3} + \frac{81}{3} = \frac{82}{3}.$$

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Problem 4. The sequence $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence (so $a_1 < a_2 < a_3 < \cdots$) bounded between 1 and 5. Which of the following must be true?

Solution. Since the sequence $\{a_n\}_n$ is bounded and increasing, its limit must be between greater than 1 but less than or equal to 5. Since a number between 1 and 5 must be greater than 0, the series $\sum_n a_n = \infty$. ◇

Problem 5. When the repeating decimal $0.\overline{12} = 0.1212121212...$ is written as a ration of integers a/b in reduced form, what is the value of $b - a$?

Solution. Note that

$$\begin{aligned}
0.\overline{12} &= 0.12 + 0.0012 + 0.000012 + \cdots \\
&= \sum_{k=1}^{\infty} \frac{12}{100^k} \\
&= \frac{12}{100} \sum_{k=0}^{\infty} \frac{1}{100^k} \\
&= \left(\frac{12}{100}\right)\left(\frac{1}{1-1/100}\right) \\
&= \left(\frac{12}{100}\right)\left(\frac{100}{99}\right) \\
&= \frac{12}{99} \\
&= \frac{3\cdot 2^2}{3^2\cdot 11} \\
&= \frac{4}{33}.
\end{aligned}$$

Thus,

$$b - a = 33 - 4 = 29.$$

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Problem 6. Evaluate

$$\int_3^{\infty} \frac{x}{(x^2-4)^2}$$

Solution. Make the substitution $u = x^2 - 4$, $du = 2x\,dx$ with $u = 3^2 - 4 = 5$, $u = \infty$. Then

$$\begin{aligned}
\int_3^\infty \frac{x}{(x^2-4)^2} &= \frac{1}{2}\int_5^\infty \frac{1}{u^2}\,du \\
&= \frac{1}{2}\lim_{u\to\infty}\left[-\frac{1}{u} + \frac{1}{10}\right] \\
&= \frac{1}{10}.
\end{aligned}$$

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Problem 7. Evaluate the integral

$$\int \sqrt{21 + 4x - x^2}\,dx.$$

Solution. First, complete the square

$$\begin{aligned}
21 + 4x - x^2 &= 12 - (x^2 - 4x) \\
&= 21 - ((x-2)^2 - 4) \\
&= 25 - (x-2)^2.
\end{aligned}$$

Now, using the trigonometric substitution $5\sin\theta = x - 2$, $5\cos\theta\,d\theta = dx$, we get

$$\begin{aligned}
\int \sqrt{21 + 4x - x^2}\,dx &= 25\int \cos^2\theta\,d\theta \\
&= \frac{25}{2}\int 1 + \cos(2\theta)\,d\theta \\
&= \frac{25}{4}(2\theta + \sin(2\theta)) \\
&= \frac{25}{4}\left[2\sin^{-1}\left(\frac{x-2}{5}\right) + \frac{2}{25}(x-2)\sqrt{25-(x-2)^2}\right] + C.
\end{aligned}$$

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Problem 8. Evaluate the integral

$$\int \frac{x^3 + x^2 + x + 4}{x^2 + 4x}\,dx.$$

Solution. First, notice that the integrand is not a proper rational function and must be reduce through long-division to a sum of a polynomial and a rational function. That is, notice that

$$x^3 + x^2 + x + 4 = (x-3)(x^2+4x) + 13x + 4.$$

Thus, the integrand becomes

$$\underbrace{x-3}_{L} + \underbrace{\frac{13x+4}{x^2+4x}}_{R}.$$

The linear part, L, we can integrate easily, but the rational function part, R, requires more work. For R, the partial fraction decomposition is of the form

$$\frac{A}{x} + \frac{B}{x+4}.$$

We can find these coefficients by the cover-up method:

$$\frac{13x+4}{x+4} = \frac{A}{x}x + \frac{B}{x+4}x, \text{ plugging in } x = 0 \implies A = 1.$$

Similarly,

$$\frac{13x+4}{x} = \frac{A}{x}(x+4) + \frac{B}{x+4}(x+4), \text{ plugging in } x = -4 \implies B = 12.$$

Thus,

$$\begin{aligned}\int \frac{x^3+x^2+x+4}{x^2+4x}\,dx &= \int L + R\,dx \\ &= \int x - 3 + \frac{1}{x} + \frac{12}{x+4}\,dx \\ &= \frac{1}{2}x^2 - 3x + \ln|x| + 12\ln|x+4| + C.\end{aligned}$$

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Problem 9. Evaluate the integral

$$\int \frac{1}{(t^2+4)^{3/2}}\,dt.$$

Solution. Using the trigonometric substitution $2\tan\theta = t$, $2\sec^2\theta\,d\theta = dt$, then $\frac{1}{2}\cos\theta = 1/\sqrt{t^2+4}$, so

$$\begin{aligned}\int \frac{1}{(t^2+4)^{3/2}}\,dt &= \int \frac{1}{8}\cos^3\theta(2\sec^2\theta)\,d\theta \\ &= \frac{1}{4}\int \cos\theta\,d\theta \\ &= \frac{1}{4}\sin\theta + C.\end{aligned}$$

$\diamond$

Problem 10. Determine whether the series converges or diverges. State which test you used, and use it to justify your answer.

$$\sum_{k=2}^{\infty} \frac{1}{k\ln k}.$$

Solution. We will show that the series diverges by the integral test. By the integral test the series converges if and only if the integral below converges

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx.$$

However, upon making the substitution $u = \ln x$, $x\,du = dx$, we get

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u}\,du = \lim_{u\to\infty}\Big[\ln u - \ln(\ln 2)\Big] = \infty.$$

$\diamond$