MA 162 Notes 10
July 16, 2019

Lecture

Recall that given a function f of x which is n-times differentiable the n-th order Taylor polynomial p_n of f centered at a is given by the equation

$$\begin{aligned} p_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}}{n!}(a)(x-a)^n. \end{aligned}$$

The Taylor polynomial provides a good approximation to the original function from which it was constructed. But how good is this approximation? To answer this question we define the *remainder* R_n of a Taylor polynomial p_n of f as

$$R_n(x) = f(x) - p_n(x) \tag{1}$$

Finding an exact value for the remainder is computationally difficult for several transcendental functions like sin, cos, tan, e, etc. but thankfully there is a result which gives a bound on the remainder; this is the so-called Taylor's remainder theorem.

Theorem 1 (Taylor's remainder theorem)**.** *Let n be a fixed positive integer. Suppose there exists a number M such that $|f^{(n+1)l(c)}| \leq M$, for all c between a and x inclusive. The remainder in the n-th order Taylor polynomial for f centered at a satisfies*

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}. \tag{2}$$

Example 2. Consider the remainder for the n-th Taylor polynomial of $\cos x$ centered at $x = 0$. By Eq. (**??**) the remainder is bounded by

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(c)| \leq M$. Note that $|f^{(n+1)}(c)|$ is either $\sin c$ or $\cos c$ (depending on whether n is even or odd) and the values of sin (or cos) between 0 and x

are bounded by 1, so a bound for the remainder is

$$|R_n(x)| \le \frac{|x-a|^{n+1}}{(n+1)!},$$

What does this mean? If we approximate, e.g. cos(0.1) using the 10-th order Taylor polynomial of $\cos x$, then the remainder

$$|R_{10}(0.1)| \le \frac{0.1^{11}}{11!} = 2.5 \times 10^{19},$$

meaning our approximation is good up to 19 decimal places, which is very good. This is in fact how calculator approximate values of sin, e^x, arctan, etc.

Example 3. Consider now approximating $\sqrt{18}$ by the 3-rd order Taylor polynomial of $f(x) = \sqrt{x}$ centered at $x = 16$. The polynomial was

$$p_3(x) = 4 + \tfrac{1}{8}(x-16) - \tfrac{1}{512}(x-16)^2 + \tfrac{1}{16\,384}(x-16)^3.$$

In the book, we determined the error in approximating $\sqrt{18}$ by $p_3(18)$ by computing $\sqrt{18}$ explicitly and taking the absolute value of its difference with $p_3(18)$; this turned out to be 3.5×10^{-5}.

We will do the same but use the remainder theorem to find an upper bound for the error. That is, in Eq. (**??**) we must find an appropriate value for M which bounds $|f^{(4)}(c)|$ for $16 \le c \le 18$. Taking the absolute value of fourth derivative of f, we have

$$|f^{(4)}(c)| = \frac{15}{16c^{7/2}}.$$

Since this is a strictly decreasing function, the maximum of $|f^{(4)}(c)|$ within $16 \le c \le 18$ must occur at $c = 16$ so $|f^{(4)}(c)| \le 5.7 \times 10^{-5}$. Thus,

$$|R_3(18)| \le \frac{M}{4!}(18-16)^4 \approx 5.7 \times 10^{-5}(2/3) \approx 3.8 \times 10^{-5},$$

which is good since the exact error is less than the bound on the error.

Example 4. Next we estimate the error of approximating $e^{0.45}$ using the sixth order Taylor polynomial for $f(x) = e^x$ centered at 0. By the remainder theorem,

$$|R_6| \le M\frac{|0.45-0|^7}{7!} \approx 7.4 \times 10^{-7}M,$$

where $|f^{(7)}(c)| = e^c \leq M$ for any $0 \leq c \leq 0.45$. Since e^c is an increasing function, the maximum is achieved at the endpoint $e^{0.45}$, but this will not do as it is exactly the value we are trying to compute.

So to obtain a bound we observe that $e^{0.45} < e^{0.2} < 4^{0.5} = 2$. Therefore,

$$|R_6| \leq M \frac{|0.45 - 0|^7}{7!} < 1.5 \times 10^{-6}.$$

Using the sixth order Taylor polynomial for e^x, we approximate $e^{0.45}$

$$p_6(0.45) = \sum_{k=0}^{6} \frac{0.45^k}{k!} \approx 1.5683114$$

which is bigger or smaller than the actual value by no more than 1.5×10^{-6}.

Example 5. Let us look at one last example. The n-th order Taylor polynomial for $f(x) = \ln(1 - x)$ centered at 0 is

$$p_n(x) = -\sum_{k=1}^{n} \frac{x^k}{k} = -x\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \cdots - \tfrac{1}{n}x^n.$$

We will determine how many terms of the Taylor polynomial are needed to approximate $f(x) = (1 - x)$ with an error less than 10^{-7} on $-1/2 \leq x \leq 1/2$.

First, we need to find a bound on $|R_n|$ using the remainder theorem. Differentiating $f(x)$ $(n + 1)$-times we get

$$|f^{(n+1)}(x)| = \frac{n!}{(1 - x)^{n+1}}.$$

On the interval $-1/2 \leq x \leq 1/2$, the maximum value $|f^{(n+1)}(x)|$ can take happens at $x = 1/2$ as the function is increasing from left to right. Thus,

$$|f^{(n+1)}(1/2)| = \frac{n!}{(1/2)^{n+1}}$$

so

$$\begin{aligned} |R_n(x)| &\leq M \frac{|x|^{n+1}}{(n+1)!} \\ &= \frac{n!}{(1/2)^{n+1}} \frac{(1/2)^{n+1}}{(n+1)!} \\ &= \frac{1}{n+1}. \end{aligned}$$

To ensure that the error by the Taylor polynomial approximation is less that 10^{-3} for $-1/2 \leq x \leq 1/2$, we need

$$|R_n| \leq \frac{1}{n+1} < 10^{-3}$$

or $n > 999$, which is huge. The error is likely o be significantly less if x is closer to 0, and therefore, you would need less terms for the approximation.