MA 162 Lecture 4
June 20, 2019

Lecture

Today we will talk about several methods for evaluating integrals of trigonometric functions. A good grasp of these techniques, together with trigonometric substitutions which are covered in tomorrow's lecture, will allow us to deal with more complicated integrals.

Powers of sine or cosine

There are two major techniques for evaluating integrals of the form $\int \sin x\, dx$ and $\int \cos^n x\, dx$. Both of these strategies will require us to know some trigonometric identities. For convenience, let us list them here:

$$\cos^2 x + \sin^2 x = 1 \tag{A}$$
$$1 + \tan^2 x = \sec^2 x \tag{B}$$
$$\sin^2 x = \frac{1 - \cos(2x)}{2} \tag{C}$$
$$\cos^2 x = \frac{1 + \cos(2x)}{2} \tag{D}$$

Example 1. Evaluate the following integrals

$$\text{(a)} \int \cos^5 dx, \qquad \text{(b)} \int \cos^4 x\, dx.$$

Solution. (a) For integrals involving *odd* powers of $\cos x$, the most effective method is to factor a power of $\cos x$ from the product and rewrite the product by using Identity (A). After doing that, the substitution $u = \sin x$,

$du = \cos x\, dx$ will allow us to evaluate the integral, as we will now see:

$$\begin{aligned}
\int \cos^5 x\, dx &= \int (\cos^4 x) \cos x\, dx \\
&= \int (1 - \sin^2 x)^2 \cos x\, dx \\
&= \int (1 - u^2)^2\, du \\
&= \int 1 - 2u^2 + u^4\, du \\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C \\
&= \underline{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C}.
\end{aligned}$$

This would also work if you replace $\cos x$ by $\sin x$ by making the necessary adjustment. (But be careful as this time since $u = \cos x$ you get $du = -\sin x\, dx$.)

(b) For integrals involving *even* powers of $\cos x$, we will apply Identity (D) *recursively*, as we will now see:

$$\begin{aligned}
\int \cos^4 dx &= \int \left(\frac{1 + \cos(2x)}{2} \right)^2 dx \\
&= \frac{1}{4} \left[\int 1 + 2\cos(2x) + \cos^2(2x)\, dx \right].
\end{aligned}$$

Because we have another even power of cosine, $\cos^2(2x)$, we need to apply Identity (D) again and we do this until we no longer get an even power of cosine. Luckily for us, we need only do this once more:

$$\begin{aligned}
\int \cos^4 dx &= \frac{1}{4} \left[\int 1 + 2\cos(2x) + \cos^2(2x)\, dx \right] \\
&= \frac{1}{4} \left[\int 1 + 2\cos(2x) + \left(\frac{1 + \cos(4x)}{2} \right) dx \right] \\
&= \frac{1}{4} \left[\int \frac{3}{2} + 2\cos(2x) + \frac{1}{2}\cos(4x)\, dx \right] \\
&= \underline{\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}.
\end{aligned}$$

The same general method works for even powers of sine, but with the recursive application of Identity (C). $\diamond$

Integrating products of powers of sine and cosine

Next we consider integrals of the form $\int \sin^m x \cos^n x \, dx$. The methods for dealing with these are summarized below.

SC1. If m is odd and positive, n real – split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x, du = -\sin x \, dx$.

SC2. If n is odd and positive, m is real – split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and use $u = \sin x, du = \cos x \, dx$.

SC3. If m and n are both even nonnegative integers – Use half-angle formulas (Identities (C) or (D)) to transform the integrand into a polynomials of $\cos(2x)$, and apply this strategy again to powers of $\cos(2x)$ greater than 1.

It is best to illustrate all of this with an example.

Example 2. Evaluate integrals

$$\text{(a)} \int \sin^4 x \cos^2 x \, dx, \qquad \text{(b)} \int \sin^3 x \cos^{-2} x \, dx.$$

Solution. (a) Using the methods described above, we want to use one of the identities (C) or (D) to half the power of the sine or cosine factors. Since the power of the cosine factors is 2, it would be faster to start there:

$$\begin{aligned}\int \sin^4 x \cos^2 x \, dx &= \int \left(\frac{1-\cos(2x)}{2}\right)^2 \left(\frac{1+\cos(2x)}{2}\right) \\ &= \frac{1}{8}\left[\int 1 - \cos(2x) - \cos^2(2x) - \cos^2(2x)\right. \\ &\qquad \left. + \cos^3(2x) \, dx\right].\end{aligned}$$

Now that we have simple powers of $\cos x$, we can apply the methods from the last section to individually evaluate these (the $1 - \cos(2x)$ can be dealt with immediately). That is,

$$\int \cos 2x = \frac{1}{2}\sin(2x),$$

$$\int \cos^2(2x) = \int \frac{1+\cos(4x)}{2}\,dx$$

$$= \frac{1}{2}x + \frac{1}{8}\sin(4x),$$

$$\int \cos^3(2x)\,dx = \int (1-\sin^2(2x))\cos(2x)\,dx$$

make the substitution $u = \sin(2x)$, $du = 2\cos(2x)\,dx$

$$= \frac{1}{2}\int (1-u^2)du$$

$$= \frac{1}{2}u - \frac{1}{6}u^3$$

$$= \underline{\frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x)}.$$

Putting all of this together, and simplifying, we get

$$\int \sin^4 x \cos^2 x\,dx = \frac{1}{16}x - \frac{1}{64}\sin(4x) - \frac{1}{48}\sin^3(2x) + C.$$

(b) This one is a tad bit easier. If at least one of the powers is a positive odd power, then this approach works:

$$\int \sin^3 x \cos^{-2} x\,dx = \int \sin^2 x \cos^{-2} x \sin x\,dx$$

by Identity (A)

$$= \int (1-\cos^2 x)\cos^{-2} x \sin x\,dx$$

now make the substitution $u = \cos x, du = -\sin x\,dx$

$$= \int -(1-u^2)u^{-2}\,du$$

$$= \int (1-u^{-2})\,du$$

$$= u + \frac{1}{u} + C$$

$$= \underline{\cos x + \frac{1}{\cos x} + C}.$$

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Remark 3. What strategy would you use to evaluate $\int \sin^{-3/2} x \cos^3 x \, dx$?

Solution. At least one of the factors, cosine in this case, has positive odd power so we can use the second method (the one we used on part (b) of Example 2) to evaluate this integral. Coincidentally, here are the details:

$$\begin{aligned}\int \sin^{-3/2} x \cos^3 x \, dx &= \int \sin^{-3/2} x \cos^2 x \cos x \, dx \\ &= \int \sin^{-3/2} x (1 - \sin^2 x) \cos x \, dx\end{aligned}$$

make the substitution $u = \sin x, du = \cos x \, dx$

$$\begin{aligned}&= \int u^{-3/2}(1 - u^2) \, du \\ &= \int u^{-3/2} - u^{1/2} \, du \\ &= -2u^{-1/2} - \frac{3}{2}u^{3/2} + C \\ &= \underline{-2 \sin^{-1/2} x - \frac{2}{3} \sin^{3/2} x + C}.\end{aligned}$$

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Reduction formulas

The previous methods are not only tedious, but fail to cover other trigonometric functions such as tangent, secant, cotangent, and cosecant (at least peripherally). We will henceforth use the following reduction formulas to deal with powers of some of the aforementioned powers of trigonometric functions as well as methods from the second section to deal with all of the remaining cases.

Reduction formulas

1. $\displaystyle\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$
2. $\displaystyle\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$
3. $\displaystyle\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$, for $n \neq 1$

4. $\sec^n x\,dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} \int \sec^{n-2} x\,dx$, for $n \neq 1$.

To deal with the case when $n = 1$ for 3 and 4 above we need more sophisticated techniques (which we will soon see). For now, take the following for granted:

$$\int \tan x\,dx = -\ln|\cos x| + C, \tag{1}$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C, \tag{2}$$

$$\int \cot x\,dx = \ln|\sin x| + C, \tag{3}$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C. \tag{4}$$

Here's an example of how you might use these formulas.

Example 4. Evaluate $\int \tan^4 x\,dx$.

Solution. Using reduction formula 3, we get:

$$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx$$

use it again on $\int \tan^2 x\,dx$

$$= \frac{1}{3}\tan^3 x - \left(\tan x - \int 1\,dx\right)$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C.$$

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Recitation

Let's talk about some of the pumping problems you will see in tonight's homework. For many of these problems, you will need to make geometrical observations to help you come up with a formula for the work required to pump out a certain amount of liquid in the tank a certain height above the tank (usually, to the top of the tank).

There are three containers you will encounter

i. cylinder,

ii. inverted cone,

iii. trapezoidal trough.

We are going to determine the forces and the work required to pump a liquid of density ρ from the tank to the top of the container.

We will use the following figure of a cylinder to guide us through our calculations:

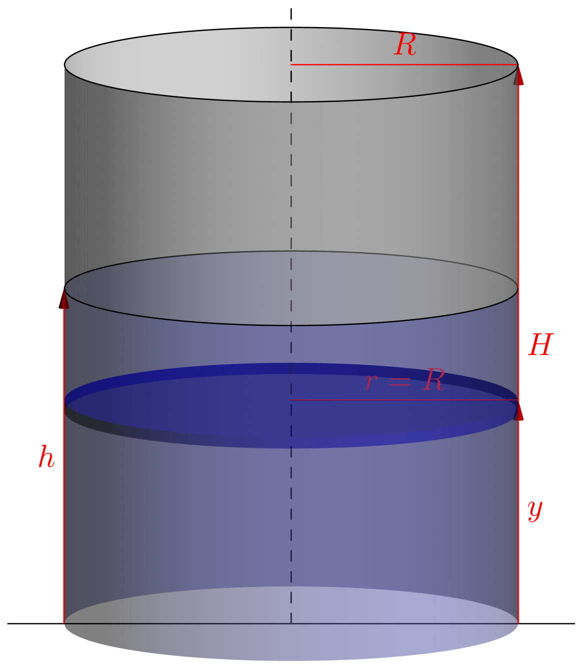

Figure 1: A cylindrical container.

The first thin we must do is determine the volume in an infinitesimal sliver (the dark blue disk in Fig. 1). The sliver is cylindrical with constant radius equal to R, so

$$dV = \pi R^2 \, dy.$$

From this formula, we can calculate the (infinitesimal) force needed to pump the sliver up (this is equivalent to the weight of the sliver):

$$dF = \pi \rho g R^2 \, dy.$$

Thus, by the equation for work, the (infinitesimal) work required to pump the sliver out of the container

$$dW = \pi \rho g R^2 (H - y) \, dy.$$

Thus, the work required to pump the whole fluid out of the container is

$$W = \int_0^h \pi \rho g R^2 (H - y)\, dy. \tag{5}$$

For the cone, consider the following image:

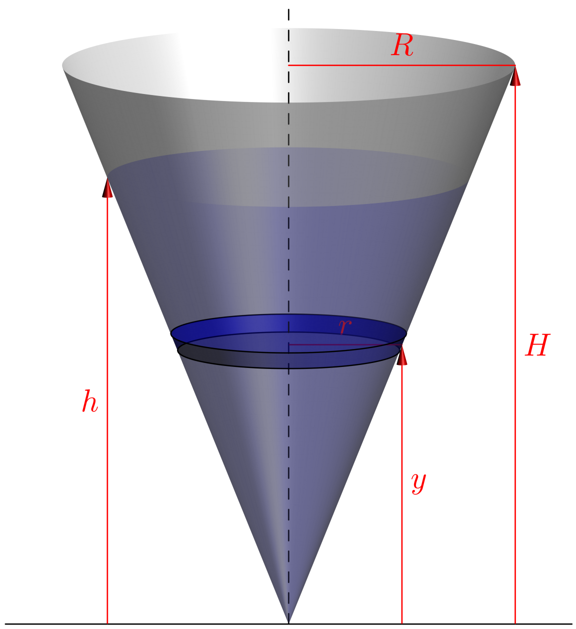

Figure 2: A inverted conical tank.

We need to determine the volume of the infinitesimal slivers (marked in dark blue in Fig. 2). The slivers are again cylindrical, but the radius r varies with respect to y

$$dV = \pi r(y)^2\, dy.$$

By using the symmetry inherent to the cylinder, we can see that the inscribed triangle of base R and height H is similar to the inscribed triangle of base r and height y so

$$\frac{y}{r} = \frac{H}{R}.$$

Thus, $r = \frac{R}{H} y$ so the infinitesimal volume can be written entirely in terms of y, like so

$$dV = \pi \left(\frac{R}{H}\right)^2 y^2\, dy.$$

Therefore, the force necessary to hold this infinitesimal volume is

$$dF = \pi\rho g\left(\frac{R}{H}\right)^2 y^2\,dy. \tag{6}$$

By the work equation, the infinitesimal work done in lifting this sliver to the top of the container is

$$dW = dF(H-y) = \pi\rho g\left(\frac{R}{H}\right)^2 y^2(H-y)\,dy.$$

Integrating this from the bottom of the container to where the liquid stops gives us the total work:

$$W = \int_0^h \pi\rho g\left(\frac{R}{H}\right)^2 y^2(H-y)\,dy. \tag{7}$$

The last of these is the trapezoidal trough which looks like the following:

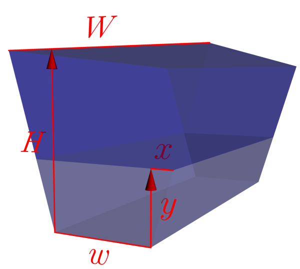

Figure 3: A trapezoidal trough.

To find the infinitesimal volume, we note first that the cross sections are rectangular, so

$$dV = L(w + 2x(y))\,dy$$

By similar triangles, we see that

$$\frac{H}{(W-w)/2} = \frac{y}{x}$$

so

$$x = \left(\frac{W-w}{2H}\right)y$$

and therefore

$$dV = L\left(w + \frac{W - w}{H}y\right)dy$$

In some of the homework questions, you will need to know the force on the face of the trapezoid, which is the integral of

$$dF = \rho g L\left(w + \frac{W - w}{H}y\right)dy \tag{8}$$

Jumping to the punch line,

$$W = \int_0^h \rho g L\left(w + \frac{W - w}{H}y\right)(L - y)\,dy. \tag{9}$$