MA 162 Lecture 5
June 25, 2019

Lecture

Today we will review for Exam 1. Because this is a summer course, Exam 1 will not necessarily cover every topic its semesterly counterpart would. So keep that in mind as you are studying for the exam.

The best way to review for an exam is to try to do problems and review concepts as you see fit. That's the style we will employ in these notes.

Vectors

Problem 1 (Spring 2018, # 2)**.** If $\mathbf{a} = \langle 2, 1, -3\rangle$ and $\mathbf{b} = \langle 1, 2, -2\rangle$, find $|5\mathbf{b} - 2\mathbf{a}|$.

Solution. The basics of vector addition and multiplication tells us that

$$\begin{aligned} 5\mathbf{b} &= \langle 5, 10, -10\rangle, \\ -2\mathbf{a} &= \langle -4, -2, 6\rangle, \\ 5\mathbf{b} - 2\mathbf{a} &= \langle 5, 10, -10\rangle + \langle -4, -2, 6\rangle \\ &= \langle 1, 8, -4\rangle. \end{aligned}$$

Therefore,

$$|5\mathbf{b} - 2\mathbf{a}| = \sqrt{1^2 + 8^2 + (-4)^2} = \sqrt{1 + 64 + 16} = \sqrt{81} = 9.$$

$\diamond$

Dot product

Problem 2 (Spring 2019, # 2)**.** Let $\mathbf{v} = \langle 2, 3, 6\rangle$ and $\mathbf{w} = \langle 1, -2, 3\rangle$. Find $\operatorname{proj}_{\mathbf{v}} \mathbf{w}$, the projection of $\mathbf{w}$ onto $\mathbf{v}$.

Solution. Recall that

$$\operatorname{proj}_{\mathbf{v}} \mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}. \tag{1}$$

Therefore,

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}} \mathbf{w} &= \operatorname{proj}_{\langle 2,3,6\rangle} \langle 1,-2,3\rangle \\
&= \left(\frac{\langle 2,3,6\rangle \cdot \langle 1,-2,3\rangle}{\langle 2,3,6\rangle \cdot \langle 2,3,6\rangle}\right)\langle 2,3,6\rangle \\
&= \left(\frac{2-6+18}{4+9+36}\right)\langle 2,3,6\rangle \\
&= \frac{14}{49}\langle 2,3,6\rangle \\
&= \frac{2}{7}\langle 2,3,6\rangle \\
&= \left\langle \frac{4}{7}, \frac{6}{7}, \frac{12}{7} \right\rangle.
\end{aligned}$$

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Problem 3 (Spring 2018, # 3)**.** Find values of x such that $\langle 4x, x, 3\rangle$ and $\langle 2, x, 5\rangle$ are orthogonal.

Solution. Recall that two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if the angle between them is $\pi/2$ (or 90°). By the definition of the dot product,

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta. \tag{2}$$

this means that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

Thus, we need to find x such that

$$\langle 4x, x, 3\rangle \cdot \langle 2, x, 5\rangle = 8x + x^2 + 15 = 0.$$

Luckily, this polynomial easily factors into

$$(x+5)(x+3) = 0.$$

Therefore, $x = -5$ or $x = -3$. ◇

Problem 4 (Spring 2018, # 4)**.** If $\mathbf{a} = 3\mathbf{i} - 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j}$, what is $\operatorname{scal}_{\mathbf{b}} \mathbf{a}$ (the scalar projection of $\mathbf{a}$ onto $\mathbf{b}$)?

Solution. Recall that, by definition, the scalar product $\operatorname{scal}_{\mathbf{v}} \mathbf{w}$ is the constant in the projection formula

$$\operatorname{proj}_{\mathbf{v}} \mathbf{w} = (\operatorname{scal}_{\mathbf{v}} \mathbf{w})\hat{\mathbf{v}},$$

where $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ is the unit vector of $\mathbf{v}$, i.e.

$$\operatorname{scal}_{\mathbf{v}} \mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|}. \tag{3}$$

Therefore,

$$\begin{aligned}
\operatorname{scal}_{\mathbf{b}} \mathbf{a} &= \frac{(\mathbf{i} - \mathbf{j}) \cdot (3\mathbf{i} - 3\mathbf{k})}{\sqrt{1^2 + 0^2 + (-1)^2}} \\
&= \frac{3}{\sqrt{2}} \\
&= \frac{3}{\sqrt{2}}.
\end{aligned}$$

$\diamond$

Problem 5 (Fall 2018, # 2)**.** Given $|\mathbf{u}| = 3$ and $|\mathbf{v}| = 4$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $2\pi/3$, find $|\mathbf{u} + \mathbf{v}|$.

Solution. By Eq. (2),

$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}||\mathbf{v}|$$

so

$$|\mathbf{u} + \mathbf{v}| = \sqrt{(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})} = \sqrt{\mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}}.$$

Thus,

$$\begin{aligned}
|\mathbf{u} + \mathbf{v}| &= \sqrt{3^2 + 2(3)(4)\cos(2\pi/3) + 4^2} \\
&= \sqrt{9 + 12 + 16} \\
&= \sqrt{37}.
\end{aligned}$$

$\diamond$

Problem 6 (Fall 2018, # 4)**.** If $\mathbf{a} = \langle 1, 1, 1 \rangle$ then the vector projection of $\langle 1, 0, 1 \rangle$ onto $\mathbf{a}$ is

Solution. By Eq. (1),

$$\begin{aligned}
\operatorname{proj}_{\mathbf{a}} \langle 1, 0, 1 \rangle &= \left(\frac{\langle 1, 1, 1 \rangle \cdot \langle 1, 0, 1 \rangle}{\langle 1, 1, 1 \rangle \cdot \langle 1, 1, 1 \rangle} \right) \langle 1, 1, 1 \rangle \\
&= \frac{2}{3} \langle 1, 1, 1 \rangle \\
&= \left\langle \frac{2}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle.
\end{aligned}$$

$\diamond$

Problem 7 (Fall 2018, # 5)**.** The angle between $\mathbf{a} = \langle 2, -1, 2\rangle$ and $\mathbf{b} = \langle 1, -1, 0\rangle$ is

Solution. By Eq. (2),

$$\langle 2, -1, 2\rangle \cdot \langle 1, -1, 0\rangle = 3 = 3\sqrt{2}\cos\theta.$$

Thus, $\cos\theta = 1/\sqrt{2}$ so $\theta = \pi/4$. $\diamond$

Problem 8 (Fall 2018, # 6)**.** Find the volume of the parallelepiped with adjacent edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ where $P(-2, 1, 0)$, $Q(2, 3, 2)$, $R(1, 4, -1)$ and $S(3, 6, 1)$ are points in $\mathbb{R}^3$.

Solution. The volume of the paralellepiped is the absolute value of the triple product

$$|\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})|. \tag{4}$$

Thus,

$$\begin{aligned}
\overrightarrow{PQ} &= (2, 3, 2) - (-2, 1, 0) = \langle 4, 2, 2\rangle \\
\overrightarrow{PR} &= (1, 4, -1) - (-2, 1, 0) = \langle 3, 3, -1\rangle \\
\overrightarrow{PS} &= (3, 6, 1) - (-2, 1, 0) = \langle 5, 5, 1\rangle \\
\overrightarrow{PR} \times \overrightarrow{PS} &= \begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix}\mathbf{k} \\
&= 8\mathbf{i} - 8\mathbf{j} + 0\mathbf{k}.
\end{aligned}$$

Thus,

$$|\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})| = \langle 4, 2, 2\rangle \cdot \langle 8, -8, 0\rangle = 16.$$

$\diamond$

Cross product

Problem 9 (Spring 2018, # 5)**.** If $\mathbf{a} = \langle 2, 0, 1\rangle$, $\mathbf{b} = \langle 3, 1, 1\rangle$, and $\mathbf{c} = \mathbf{a} \times \mathbf{b} = \langle c_1, c_2, c_3\rangle$, what is c_2?

Solution. By the cross product formula

$$c_2 = -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -(2 - 3) = 1.$$

$\diamond$

Problem 10 (Spring 2018, # 6)**.** Find the area of the parallelogram with vertices $A(0,0,0)$, $B(1,0,1)$, $C(2,1,0)$, and $D(3,1,1)$.

Solution. Note that since $C(2,1,0) = B(1,0,1) + D(3,1,1)$, the points do indeed form a parallelogram. Now, recall that area of the parallelogram is given by the magnitude of the cross product of the vector emanating from a vertex of the parallelogram. Pick a base point, say $A(0,0,0)$. Then

$$\begin{aligned}
\overrightarrow{AB} &= (1,0,1) - (0,0,0) = \langle 1,0,1\rangle, \\
\overrightarrow{AC} &= (2,1,0) - (0,0,0) = \langle 2,1,0\rangle, \\
|\overrightarrow{AB} \times \overrightarrow{AC}| &= \left| \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \mathbf{k} \right| \\
&= |-\mathbf{i} + 2\mathbf{j} + \mathbf{k}| \\
&= \sqrt{(-1)^2 + 2^2 + 1^2} \\
&= 2.
\end{aligned}$$

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Problem 11 (Fall 2018, # 1)**.** The area of a triangle with vertices $(1,1,1)$, $(2,2,3)$ and $(1,2,2)$ is

Solution. To find the area of the triangle, we first find the area of the corresponding parallelepiped and half its area. That is,

$$\begin{aligned}
(2,2,3) - (1,1,1) &= \langle 1,1,2\rangle, \\
(1,2,2) - (1,1,1) &= \langle 0,1,1\rangle, \\
\langle 1,1,2\rangle \times \langle 0,1,1\rangle &= \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\
&= \langle -1,-1,-1\rangle, \\
\frac{1}{2}|\langle 1,1,2\rangle \times \langle 0,1,1\rangle| &= \frac{\sqrt{3}}{2}.
\end{aligned}$$

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Problem 12 (Fall 2018, # 3)**.** Give $|\mathbf{u}| = 1$, $|\mathbf{v}| = 5$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/6$, find $|\mathbf{u} \times \mathbf{v}|$

Solution. By the cross product formula,

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta, \tag{5}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Therefore,

$$|\mathbf{u} \times \mathbf{v}| = 5 \sin(\pi/6) = \frac{5}{2}.$$

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Area between curves

Problem 13 (Spring 2018, # 7)**.** Find the area of the region bounded by $y = x^2 - 4x$ and $y = -2x$.

Solution. First, we need to find the endpoints of the region in terms of x. These are the points where the y-values and x-values of the curves coincide:

$$\begin{aligned} x^2 - 4x &= -2x \\ x^2 - 2x &= 0 \\ x(x-2) &= 0, \end{aligned}$$

so $x = 0, x = 2$.

Thus, the area under the curve is

$$\begin{aligned} \int_0^2 -2x - (x^2 - 4x)\, dx &= \int_0^2 2x - x^2\, dx \\ &= \left[x^2 - \frac{1}{3}x^3 \right]_0^2 \\ &= 4 - \frac{8}{3} \\ &= \frac{4}{3}. \end{aligned}$$

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Problem 14 (Spring 2018, # 8)**.** Find the area of the region bounded by $y = x$ and $x = y^2 - 2y$.

Solution. Since x is a function of y, it is best to integrate with respect to y. Note, therefore, that $x = y$ is above $x = y^2 - 2y$ and the two curves intersect at $y = 0$ and $y = 3$ since $0 = y^2 - 3y = y(y-3)$.

Thus, the area of the bounded region is

$$\begin{aligned}\int_0^3 3y - y^2\,dy &= \left[\frac{3}{2}y^2 - \frac{1}{3}y^3\right]_0^3 \\ &= \frac{27}{2} - \frac{27}{3} \\ &= \frac{27}{6} \\ &= \frac{9}{2}.\end{aligned}$$

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Problem 15 (Fall 2018, # 7)**.** The area of the region between the curves $y = x^3$ and $y = 4x$ is

Solution. Note that $4x$ is above x^3 and the two curves meet when $x = 0$, $x = 2$, $x = -2$. If you draw the graph, this region has two components. Therefore, the area bounded by the region is the sum of

$$\begin{aligned}\int_0^2 4x - x^3\,dx &= \left[2x^2 - \frac{1}{4}x^4\right]_0^2 \\ &= 8 - 4 \\ &= 4, \\ \int_0^2 x^3 - 4x\,dx &= \left[\frac{1}{4}x^4 - 2x^2\right]_{-2}^0 \\ &= 4.\end{aligned}$$

Thus, the area of the bounded region is 8. ◇

Solids of revolution

Problem 16 (Spring 2018, # 9)**.** The area between the graphs of $y = x^2$ and $y = 2x$ is revolved around the x-axis. If the disk/washer method is used, the integral representing the volume of the resulting solid is

Solution. By the washer method,

$$A(x) = \pi[R(x)^2 - r(x)^2]. \tag{6}$$

In this case, $R(x) = 2x$ and $r(x) = x^2$ and we integrate from $x = 0$ to $x = 2$, as follows:

$$\begin{aligned} V &= \int_0^2 \pi[(2x)^2 - (x^2)^2]\,dx \\ &= \int_0^2 \pi(4x^2 - x^4)\,dx \\ &= \pi\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 \\ &= \pi\left(\frac{32}{3} - \frac{32}{5}\right) \\ &= \frac{64}{15}\pi. \end{aligned}$$

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Problem 17 (Spring 2018, # 10)**.** The area between the graphs of $y = x^2$ and $y = 2x$ is revolved around the x-axis. If the method of cylindrical shells is used, the integral representing the volume of the resulting solid is

Solution. Using the method of cylindrical shells, we must find

$$V = \int 2\pi r(y)h(y)\,dy. \tag{7}$$

In this case, we know the formula is in terms of y because the radius, which must be perpendicular to the x-axis is exactly $r(y) = y$. Now, given a y, we must find the height which is how far along the x-direction we can stretch. In this case, the furthest left is $y/2$ and the furthest right is $\sqrt{y}$ so

$$\int_0^4 2\pi\left(\sqrt{y} - \frac{y}{2}\right) dy.$$

(Note that the endpoints of the region are the same as those for the previous problem, namely $(0, 0)$ and $(2, 4)$, except we are taking the y-components.) ◇

Problem 18 (Spring 2018, # 11)**.** What value of the positive constant c makes the volume of the solid obtained by rotating the area between $x = 0$, $y = e^{2x}$, and $x = c$ about the x-axis equal to π?

Solution. The best method to employ is the disk method and that will give us a volume of

$$\int_0^c \pi(e^{2x})^2\,dx = \frac{\pi}{4}(e^{4c} - 1)$$

so we must have $(e^{4c} - 1) = 4$. Therefore,

$$\begin{aligned} e^{4c} - 1 &= 4 \\ e^{4c} &= 5 \\ 4c &= \ln 5 \\ c &= \frac{1}{4} \ln 5. \end{aligned}$$

$\diamond$

Problem 19 (Spring 2018, # 12)**.** An unusual pyramid of height 2 is sitting on the xy-plane. If the cross-section at a level $z \geq 0$ is a square of side $\sqrt{2-z}$, what is the volume of this strange pyramid?

Solution. This might look a bit tricky at first, but if you try to draw the cross sections it is clear that the volume is

$$\int_0^2 (\sqrt{2-z})^2 \, dz = 2.$$

$\diamond$

Problem 20 (Fall 2018, # 8)**.** Find the volume of the region bounded by the curves $y = -x$ and $y = -x^2$ rotated about the x-axis

Solution. The two curves intersect at $x = 0$ and $x = 1$. By the washer method,

$$\begin{aligned} V &= \int_0^1 \pi[(-x)^2 - (-x^2)^2] \, dx \\ &= \int_0^1 \pi(x^2 - x^4) \, dx \\ &= \pi\left(\frac{1}{3} - \frac{1}{15}\right) \\ &= \frac{2}{15}\pi. \end{aligned}$$

$\diamond$

Problem 21 (Fall 2018, # 9)**.** Using the method of cylindrical shells, find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = x$ about $x = 1$.

Solution. After sketching the graph of the region in question, it is evident that $r(x) = 1 - x$ and $h(x) = x - x^2$ so

$$\int_0^1 2\pi(1-x)(x-x^2)\,dx = \int_0^1 2\pi x(1-x)(1-x)\,dx = \int_0^1 2\pi x(1-x)^2\,dx.$$

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Problem 22 (Fall 2018, # 10)**.** The base of a solid in the xy-plane is bounded by $y = 4 - x^2$ and $y = 0$. Its cross-sections perpendicular to the y-axis are squares. Find the volume of the solid:

Solution. The cross sections, if you draw them, have length $2\sqrt{4-y}$ so

$$V = \int_0^4 (2\sqrt{4-y})^2\,dy = \int_0^4 4(4-y)\,dy.$$

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Applications of integration

Problem 23 (Fall 2018, # 11)**.** A force of 4 lbs stretches a spring with a natural length of 12 in to 18 in. Find the total work done by stretching the spring from a length of 18 in to 24 in. Note: 1 ft = 12 in.

Solution. By Hooke's law, $F = k(x - x_0)$ where x_0 is the equilibrium (on natural) length of the spring. Therefore,

$$k = \frac{4}{18-12}\frac{12}{1} = 8 \text{ lbs/ft}.$$

Thus, the work done in stretching the spring from 18 in to 24 in is

$$\begin{aligned} W &= \int_{x_1}^{x_2} k(x-x_0)\,dx \\ &= \frac{1}{2}k([x_2-x_0]^2 - [x_1-x_0]^2) \\ &= 4\left[\left(\frac{24-12}{12}\right)^2 - \left(\frac{18-12}{12}\right)^2\right] \\ &= 4\left(\frac{3}{4}\right) \\ &= 3. \end{aligned}$$

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