MA 162 Lecture 7
June 27, 2019

Lecture

Recall that a *proper* rational function $f(x)$ is the quotient of two polynomials $P(x) = a_m x^m + \cdots + a_0$ and $Q(x) = b_n x^n + \cdots + b_0$, where $m < n$, i.e.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_m x^m + \cdots + a_0}{b_n x^n + \cdots + b_0}.$$

Today we will focus on methods for dealing with rational functions which have repeated factors and irreducible quadratic factors. We will end the lecture with short detour into long division, which will let us perform our methods on improper rational functions.

Repeated Linear Factors

Let $f(x) = P(x)/Q(x)$ be a proper rational function with linear factor $(x-r)^m$ in the denominator $Q(x)$. Then the partial fraction decomposition of $f(x)$ has a partial fraction for each power of $(x-r)$ up to and including the m^{th} power; i.e. the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}, \tag{1}$$

where $A_1, \ldots, A_m$ are constants to be determined.

Let us take a look at an example to see how this works in practice.

Example 1. Find the partial fraction decomposition of

$$f(x) = \frac{5x^2 - 3x + 2}{x^3 - 2x^2}$$

and use this to find the improper integral $\int f(x)\,dx$.

Solution. First note that the denominator $Q(x) = x^3 - 2x^2$ can be factored as $x^2(x-2)$, where x is repeated twice. Therefore, Eq. (1) tells us that the partial fraction decomposition of $f(x)$ will contain a term of the form

$$\frac{A}{x} + \frac{B}{x^2}.$$

The full partial fraction decomposition has the form

$$f(x) = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}.$$

To solve for the coefficients A, B, C, we need to set the left-hand side of the equation above to the original definition for $f(x)$ and make a few calculations. By far the quickest method to solve for the coefficients is by using the so-called Heaviside cover-up method; this is nothing more than a fancy term for plugging an appropriate value of x into our expressions to 'cover-up' the coefficients we are not interested in. For example, suppose we wanted to find the value of C first in the partial fraction decomposition of $f(x)$. Then we would first set up the equality like so

$$\frac{5x^2 - 3x + 2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

then multiply by $(x-2)$ to get

$$\frac{5x^2 - 3x + 2}{x^2} = \frac{A}{x}(x-2) + \frac{B}{x^2}(x-2) + C.$$

Now we 'cover-up' A and B by setting $x = 2$ to get

$$\begin{aligned} \frac{5x^2 - 3x + 2}{x^2} &= \frac{A}{x}(x-2) + \frac{B}{x^2}(x-2) + C \\ \frac{5 \cdot 2^2 - 3 \cdot 2 + 2}{2^2} &= \frac{A}{2}(2-2) + \frac{B}{2^2}(2-2) + C \\ \implies C = 4. \end{aligned}$$

Now we do the same for B (after making the substitution $C = 4$), but less verbosely:

$$\begin{aligned} \frac{5x^2 - 3x + 2}{x^2(x-2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{4}{x-2} \\ \frac{5x^2 - 3x + 2}{x-2} &= Ax + B + \frac{4}{x-2}x^2 \\ \frac{5 \cdot 0^2 - 3 \cdot 0 + 2}{0-2} &= A \cdot 0 + B + \frac{4}{0-2} \cdot 0 \\ \implies B = -1. \end{aligned}$$

Unfortunately (as mentioned in that Wikipedia link), the cover-up method only works for the largest power of a linear factor and so, to solve for A, we need to substitute both $B = -1$ and $C = 4$ into the original equation, like so

$$\frac{5x^2 - 3x + 2}{x^2(x-2)} = \frac{A}{x} - \frac{1}{x^2} + \frac{4}{x-2}$$

and make any reasonable choice for x (don't chose something which will result in division-by-zero), in this case $x = 1$ is the best choice since

$$\begin{aligned} \frac{5 \cdot 1^2 - 3 \cdot 1 + 2}{1^2(1-2)} &= \frac{A}{1} - \frac{1}{1^2} + \frac{4}{1-2} \\ 4 &= A - 1 - 4 \\ \implies A &= 1. \end{aligned}$$

Therefore,

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{4}{x-2}$$

and we may now easily show that

$$\begin{aligned} \int \frac{5x^2 - 3x + 2}{x^3 - 2x^2}\, dx &= \int \frac{1}{x} - \frac{1}{x^2} + \frac{4}{x-2}\, dx \\ &= \ln|x| + \frac{1}{x} + 4\ln|x+2| + C. \end{aligned}$$

◇

Irreducible quadratic factors

The last method we develop deals with irreducible quadratic factors. Recall that, by the fundamental theorem of algebra, given a polynomial like, say $Q(x)$ the denominator of the rational function $f(x)$, we can factor $Q(x)$ into a product of linear factors $x - r$ and irreducible quadratic factors $ax^2 + bx + c$ whose discriminant $b^2 - 4ac < 0$.

Now, let $f(x) = P(x)/Q(x)$ be a rational function. Suppose an irreducible factor $ax^2 + bx + c$ appears as a factor of $Q(x)$. Then the partial fraction decomposition of $f(x)$ contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}, \tag{2}$$

where A and B are unknown coefficients to be determined.

Let us work an example to see how this method is used in practice.

Example 2. Give the partial fraction decomposition of

$$\frac{10}{(x-2)^2(x^2+2x+2)}.$$

Solution. The denominator is already fully factored, Eq. (2) tells us we will have a factor of the form

$$\frac{Cx+D}{x^2+2x+2},$$

and we see that the full partial fraction decomposition must have the form

$$f(x) = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+2x+2}.$$

We can use the cover-up method to solve for just one of the variables, B, as follows

$$\begin{aligned}
\frac{10}{(x-2)^2(x^2+2x+2)} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+2x+2} \\
\frac{10}{x^2+2x+2} &= A(x-2) + B + \frac{C}{x^2+2x+2}(x-2)^2 \\
\frac{10}{2^2+2\cdot 2+2} &= A(2-2) + B + \frac{C}{2^2+22+2}(2-2)^2 \\
\implies B &= 1.
\end{aligned}$$

Now, substitute $B = 1$ into the first line of the last equation

$$\frac{10}{(x-2)^2(x^2+2x+2)} = \frac{A}{x-2} + \frac{1}{(x-2)^2} + \frac{Cx+D}{x^2+2x+2}$$

and rearrange the equation to get (an unwieldy expression which we must work to simplify)

$$\begin{aligned}
10 &= A(x-2)(x^2+2x+2) \\
&\quad + (x^2+2x+2) \\
&\quad + (Cx+D)(x-2)^2 \\
&= A(x^3-2x-4) \\
&\quad + (x^2+2x+2) \\
&\quad + (Cx+D)(x^2-4x+4) \\
&= Ax^3-2Ax-4A+x^2 \\
&\quad + 2x+2+Cx^3-4Cx^2+4Cx \\
&\quad + Dx^2-4Dx+4D \\
&= (A+C)x^3+(1-4C+D)x^2 \\
&\quad + (-2A+2-4D+4C)x+(-4A+2+4D)
\end{aligned}$$

Matching the coefficients of the polynomial on the right-hand side with the polynomial on the left-hand side, we get the following equalities

$$\begin{aligned}
A+C &= 0, \\
1-4C+D &= 0, \\
-2A+2-4D+4C &= 0, \\
-4A+2+4D &= 10.
\end{aligned}$$

Let us solve for A, C, D systematically. The top equation tells us that $A = -C$ and the second equation tells us that $D = 4C - 1$ so, by the third equation

$$\begin{aligned}
-2(-C)+2-4(4C-1) &= 0 \\
2C+2-16C+4+4C &= 0 \\
-10C &= -6 \\
\implies C &= 3/5.
\end{aligned}$$

Then, $D = 4(3/5) - 1 = 7/5$ and $A = -3/5$ so the partial fractions decomposition of $f(x)$ is

$$-\frac{\frac{3}{5}}{x-2} + \frac{1}{(x-2)^2} + \frac{\frac{3}{5}x+\frac{7}{5}}{x^2+2x+2}.$$

$\diamond$

Example 3. Evaluate the integral

$$\int \frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\,dx.$$

Solution. The partial fraction decomposition of the rational function in integrand is

$$\frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+3}.$$

Multiplying both sides by the denominator of the right-hand side

$$\begin{aligned} 7x^2 - 13x + 13 &= A(x^2-2x+3) + (Bx+C)(x-2) \\ &= (A+B)x^2 + (-2A-2B+C)x + (3A-2C). \end{aligned}$$

By the cover-up method, substituting $x = 2$ in the expression, we get

$$A(2^2 - 2\cdot 2 + 3) = 7\cdot 2^2 - 13\cdot 2 + 13$$

so $A = 5$. With this and the equations we get from matching equal powers of x,

$$\begin{aligned} A + B &= 7, \\ -2A - 2B + C &= -13, \\ 3A - 2C &= 13. \end{aligned}$$

Solving this system of equations, $B = 2$ and $C = 1$ and therefore

$$\int \frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\,dx = \int \frac{5}{x-2} + \frac{2x+1}{x^2-2x+3}\,dx.$$

The first fraction is easy enough to deal with, its integral is

$$5\ln|x-2|;$$

the difficult one is

$$\int \frac{2x+1}{x^2-2x+3}\,dx.$$

The key to solving this problem is noting that if we make the substitution $u = x^2 - 2x + 3$, $du = 2x - 2\,dx$ so if we can turn the numerator into $2x - 2$, we could solve the integral, so

$$\int \frac{2x+1}{x^2-2x+3}\,dx = \int \frac{2x-2}{x^2-2x+3}\,dx + \int \frac{3}{x^2-2x+3}\,dx.$$

The u-substitution part is easy, we get

$$\int \frac{2x-2}{x^2-2x+3}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln|x^2-2x+3|.$$

The second part requires knowledge about certain antiderivatives, in particular, we need to know that

$$\int \frac{1}{x^2+1}\,dx = \arctan x.$$

After rearranging the second integral, we have

$$\begin{aligned}\int \frac{3}{x^2-2x+3}\,dx &= 3\int \frac{1}{(x-1)^2+2}\,dx \\ &= \frac{3}{2}\int \frac{1}{(\frac{x-1}{\sqrt{2}})^2+1}\,dx\end{aligned}$$

make the u-sub $u=(x-1)/\sqrt{2}$, $\sqrt{2}du = dx$, then

$$\begin{aligned}&= \frac{3}{2}\sqrt{2}\int \frac{1}{u^2+1}\,du \\ &= \frac{3}{\sqrt{2}}\tan u \\ &= \frac{3}{\sqrt{2}}\arctan\left(\frac{x-1}{\sqrt{2}}\right).\end{aligned}$$

Therefore, the full integral is

$$5\ln|x-2| + \ln|x^2-2x+3| + \frac{3}{\sqrt{2}}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + C.$$

◇

Long division

Now, these methods work very well for proper rational functions, i.e. functions $f(x) = p(x)/q(x)$ with $\deg p < \deg q$. But what if we are given a function of the form

$$f(x) = \frac{2x^3+11x^2+28x+33}{x^2-x-6}?$$

We must do long division.

To do this, take the denominator $x^2 - x - 6$ and determine what the highest chunk of $2x^3 + 11x^2 + 28 + 33$ you can take is.

$$2x(x^2 - x - 6) = 2x^3 - 2x^2 - 12x$$

so

$$2x^3 + 11x^2 + 28x + 33 - (2x^3 - 2x^2 - 12x) = 13x^2 + 40x + 33$$

and

$$13(x^2 - x - 6) = 13x^2 - 13x - 78$$

so

$$13x^2 + 40x + 33 - (13x^2 - 13x - 78) = 53x + 111.$$

Thus,

$$f(x) = \frac{(2x + 13)(x^2 - x - 6) + 53x + 111}{x^2 - x - 6} = 2x + 13 + \frac{53x + 111}{x^2 - x - 6}$$

now f is the sum of a polynomial and a rational function, which we know how to deal with.

Recitation

Examples to do

Example 4. Evaluate

$$\int_0^1 (2x - 1)e^{2x}\, dx.$$

Solution. By tabular integration,

$$\int_0^1 (2x - 1)e^{2x}\, dx = \frac{1}{2}(2x - 1)e^{2x} - \frac{1}{2}e^{2x}\Big|_0^1 = 1.$$

◇

Example 5. Evaluate the integral

$$\int_0^{2\sqrt{2}} \frac{x^2}{\sqrt{16 - x^2}}\, dx.$$

Solution. By making the trigonometric substitution, $x = 4\sin\theta$, $dx = 4\cos\theta\, d\theta$, so the bounds change to $x = 2\sqrt{2} = 4\sin\theta \implies \theta = \pi/4$ and $x = 0 = 4\sin\theta \implies \theta = 0$. Thus

$$\begin{aligned}
\int_0^{2\sqrt{2}} \frac{x^2}{\sqrt{16-x^2}}\, dx &= \int_0^{\pi/4} \frac{(4\sin\theta)^2 4\cos\theta}{\sqrt{16-16\sin^2\theta}}\, d\theta \\
&= \int_0^{\pi/4} \frac{(16\sin^2\theta)(4\cos\theta)}{4\cos\theta}\, d\theta \\
&= \int_0^{\pi/4} 16\sin^2\theta\, d\theta \\
&= \int_0^{\pi/4} 16\left(\frac{1-\cos(2\theta)}{2}\right) d\theta \\
&= \int_0^{\pi/4} 8 - 8\cos(2\theta)\, d\theta \\
&= 8\theta - 4\sin(2\theta)\Big|_0^{\pi/4} \\
&= 8(\pi/4) - 4\sin(\pi/4) - (8(0) - 4\sin(0)) \\
&= 2\pi - 4.
\end{aligned}$$

◇

Example 6. Evaluate the integral

$$\int \cos^5\theta \sin^4\theta\, d\theta.$$

Solution. Since the cosine factor is an odd power, do the following

$$\begin{aligned}
\int \cos^5\theta \sin^4\theta\, d\theta &= \int \cos^4\theta \sin^4\theta \cos\theta\, d\theta \\
&= \int (\cos^2\theta)^2 \sin^4\theta \cos\theta\, d\theta \\
&= \int (1-\sin^2\theta)^2 \sin^4\theta \cos\theta\, d\theta
\end{aligned}$$

make the substitution $u = \sin\theta,\ du = \cos\theta\, d\theta$

$$
\begin{aligned}
&= \int (1 - u^2)^2 u^4 \, du \\
&= \int (u^4 - 2u^2 + 1) u^4 \, du \\
&= \int u^8 - 2u^6 + u^4 \, du \\
&= \frac{1}{9}u^9 - \frac{2}{7}u^6 + \frac{1}{5}u^5 + C \\
&= \frac{1}{9}\sin^9\theta - \frac{2}{7}\sin^7\theta + \frac{1}{5}\sin^5\theta + C.
\end{aligned}
$$

◇