MA 162 Lecture 8
July 2, 2019

Lecture

Definition of sequences

Recall that a *sequence* of real numbers $\{a_n\}$ is an assignment to each natural number $n = 1, 2, \ldots$ (called an *index* of the sequence) a real number a_n (called a *term*).

There are three main ways of defining a sequence.

- Through a function. For example:

$$a_n = \frac{n}{n+1}, \quad \{a_n\} = \{\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6} \ldots\}; \tag{1}$$

- Through a *recurrence relation*. For example: the Fibonacci sequence, which is defined

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + b_{n-2}, \quad \{a_n\} = \{1, 1, 2, 3, 5 \ldots\}; \tag{2}$$

- Through inference: For example, the digits of π form the sequence

$$\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots\}$$

Remark. Note that this last way of defining a sequence, through inference, is by far the least reliable as nowhere in the definition of a sequence does it say that my choices for the terms of the sequence 'have to follow a pattern.' We may very well pick numbers at random, in which case, we would not be able to infer, from the first few terms, what subsequent terms in the sequence are.

Today we will learn more about sequences. In particular, we will develop methods to determine the behavior of sequences as $n \to \infty$.

Graphing sequences

Sequences, like functions, can be graphed and we may learn much about the behavior of a sequence as $n \to \infty$ by doing this.

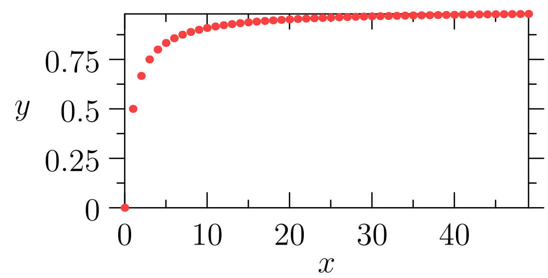

Figure 1: The first 50 terms of the sequence (1).

Fig. 1 shows the first 50 terms of the sequence given by (1). Notice that as we go further and further out into the sequence, the terms of (1) seem to be *getting closer* and closer to 1.

On the other hand, if you plot the terms of the Fibonacci sequence, (2), the terms *grow without bounds*. We will make more precise what we mean by *getting closer* and *growing without bounds*.

Definition of a limit

In the last section, we motivated the notion of a limit (a value to which a sequence gets closer and closer). Now we are going to state some terminology and say precisely what we mean by getting close to the limit.

Definition.

1. We say that $\{a_n\}$ converges to L if a_n gets as close to L as we want for sufficiently large n.

2. We say that $\{a_n\}$ converges to ∞ if a_n gets as large as we want for sufficiently large n.

3. We say that $\{a_n\}$ converges to ∞ if a_n gets as large as small as negative as we want for sufficiently large n.

A sequence which does not converges is said to *diverges*.

In all three of these cases we write

$$L = \lim_{n\to\infty} a_n$$

for the limit of $\{a_n\}$.

Remark. Traditionally, when the limit of a sequence is not *finite* we say that the sequence *diverges*. However, following the book, we will make the stipulation that the limit of a sequence can be $\pm\infty$.

The precise definition of 'as close as we want' is beyond the scope of this class, but we can say a little about this here as it is covered in the book. Intuitively, what does it mean for a sequence $\{a_n\}$ to get close to L? One possible meaning could be this, no matter what tolerance for error you have, we can find a term of the sequence such that subsequent terms match that error-tolerance, i.e. for any $\epsilon > 0$ there is an integer N such that for $n > N$

$$|L - a_n| < \epsilon.$$

For a function to converge to infinity, means that for any number $M > 0$ we can find counting number N such that for $n > N$

$$a_n > M.$$

Formally:

Definition.

1. A sequence $\{a_n\}$ converges to L if for any $\epsilon > 0$, there exists an N such that $n \geq N$ implies $|L - a_n| < \epsilon$.
2. A sequence $\{a_n\}$ converges to ∞ if for any $M > 0$, there exists an N such that $n \geq N$ implies $a > M$.
3. A sequence $\{a_n\}$ converges to $-\infty$ if for any $M < 0$, there exists an N such that $n \geq N$ implies $a < M$.

Finding limits

Now, given a sequence how can we get a hold of the limit? For sequences which are defined functionally, there is a simple way.

Theorem 1. *Let $\{a_n\}$ be a sequence such that $a_n = f(n)$ for a function f. Then, if the limit L of $f(x)$ as $x \to \infty$ exists,* $\lim_n a_n = L$.

What this theorem is telling us is that we can take the limit of a sequence by employing techniques we learned about limits of functions. That is, if we can find a function representation for the terms of the sequence, it is enough to look at the limit of the function as $x \to \infty$.

Let us look at an example of this.

Example 2. Find the limit of the sequence $\{(3n^2-1)/(10n+5n^2)\}_{n=2}^{\infty}$, if it exists.

Solution. By Theorem 1, we need only find a function $f(x)$ such that $a_n = f(n)$ and find $L = \lim_{x\to\infty} f(x)$. It is easy, from the form of the terms of the sequence, to deduce that

$$f(x) = \frac{3x^2-}{10x+5x^2}.$$

Now

$$\begin{aligned} L &= \lim_x f(x) \\ &= lim_x \frac{3x^2-1}{10x+5x^2} \end{aligned}$$

which, by L'Hôpital's rule becomes

$$= \lim_x \frac{6x^2}{10+10x}$$

which, again, by L'Hôpital's, becomes

$$\begin{aligned} &= \lim_x \frac{6}{10} \\ &= \frac{3}{5}. \end{aligned}$$

◇

Let us look at another example.

Example 3. Find the limit of the sequence $\{e^{2n}/n\}$ if it exists.

Solution. Here it is easy to see that

$$f(x) = \frac{e^{2x}}{x},$$

so

$$\begin{aligned} \lim_n \{e^{2n}/n\} &= \lim_x f(x) \\ & \lim_x \frac{e^{2x}}{x} \\ &= \lim_x 2e^{2x} \\ &= \infty. \end{aligned}$$

So the sequence converges to ∞. ◇

Here are some important properties about sequences with *finite* limits.

Theorem 4. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences with* $\lim a_n = L$ *and* $\lim b_n = K$ *both finite. Then*

1. $\lim_n a_n \pm b_n = L + K$*;*
2. $\lim_n ca_n = cL$*;*
3. $\lim_n a_n b_n = LK$*;*
4. $\lim_n a_n/b_n = L/K$*, provided* $K \neq 0$*;*
5. $\lim_n {a_n}^p = L^p$ *provided* $a_n \geq 0$*.*

Remark. Why is it important that L and K be finite in the theorem above? Consider the following sequences:

$$a_n = \frac{1}{n}, \quad b_n = n, \quad c_n = n^2.$$

By part 3 of Theorem 2, we would have

$$\lim_n a_n b_n = 0 \cdot \infty = 0,$$

but upon closer inspection,

$$\lim_n a_n b_n = \lim_n n^{-1} \cdot n =_n 1 = 1.$$

On the other hand,

$$\lim_n a_n b_n = 0 \cdot \infty = 0,$$

but

$$\lim_n a_n b_n = \lim_n n^{-1} n^2 = \lim_n n = \infty.$$

The moral of the story is that ∞ is not a proper number and if we try to force it to play that role, we may get unexpected results.

Another extremely important tool that carries over from limits of functions is the so-called *squeeze theorem*.

Theorem 5 (Squeeze theorem)**.** *Let* $\{a_n\}, \{b_n\}, \{c_n\}$ *be sequences with* $L = \lim_n a_n =_n b_n$ *and such that eventually* $a_n \leq c_n \leq b_n$*, i.e. past a certain threshold* $n > N$*, then* $\lim_n c_n = L$*.*

Corollary 6. *If* $\lim_n |a_n| = 0$ *then* $\lim_n a_n = 0$*.*

Sketch. This follows easily from the squeeze theorem, since

$$-|a_n| \le a_n \le |a_n|,$$

so $\lim_n |a_n| = 0$ and $\lim_n -|a_n| = -0 = 0$. Thus, $\lim_n a_n = 0$. □

Here is an example of how you could apply Corollary 6 to determine the convergence of a seemingly complicated sequence.

Example 7. Find the limit, if it exists, of

$$\left\{ \frac{\cos(\pi n)}{n} \right\}.$$

Solution. Note that

$$\cos(\pi n) = \begin{cases} -1 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Therefore,

$$\left\{ \left| \frac{\cos(\pi n)}{n} \right| \right\} = \left\{ \frac{1}{n} \right\},$$

whose limit is 0. Corollary 6 then tells us that

$$\lim_n \frac{\cos(\pi n)}{n} = 0.$$

◇

Geometric sequences

You will be encountering geometric series very soon, so we will say a little bit about geometric sequences here. A *geometric sequence* is a sequence of the form

$$a_n = ar^n$$

for some real numbers a, r.

The limit of a geometric sequence is very well understood. For geometric sequences with $a = 1$, the following holds.

Theorem 8. *The sequence $\{r^n\}$ converges if $-1 < r \leq 1$ and diverges for all other values of r. Also,*

$$\lim_n r^n = \begin{cases} 0 & \text{if } -1 < r < 1, \\ 1 & \text{if } r = 1. \end{cases}$$

Example 9. Determine if the following sequences converge or diverge. If the sequence converges, determine the limit.

(a) $\{(1/\sqrt{2})^{n+1}\}$

(b) $\{\sqrt{2}^n\}$

(c) $\{(-1)^n\}$

(d) $\{1^n\}$

Solution. For (a) note that the terms are geometric with $r = 1/\sqrt{2} < 1$ after we play with the exponent of the sequence, i.e. $(1/\sqrt{2})^{n+1} = (1/\sqrt{2})(1/\sqrt{2})^n$. Therefore, the sequence $\{(1/\sqrt{2})^n\}$ converges to 1 by Theorem 7, and by part 2 of Theorem 4, the limit of the original sequence is $1/\sqrt{2} = \sqrt{2}/2$.

For (b), by Theorem 7, since the terms are geometric with $r = \sqrt{2} > 1$, the sequence converges to ∞.

For (c), the sequence is geometric with $r = -1$ so the sequence diverges. Moreover, it does not converge to any finite or infinite value.

For (d), the sequence is geometric with $r = 1$ so the limit is 1; but you, very likely, already saw that by writing out the first few terms. $\diamond$

Monotonicity

There are two more important properties of sequences which will let us determine limits. The first of these is the concept of monotonicity.

Definition 10. Let $\{a_n\}$ be a sequence.

1. We say $\{a_n\}$ is increasing if $a_{n+1} > a_n$ for every n.

2. We say $\{a_n\}$ is decreasing if $a_{n+1} < a_n$ for every n.

3. If $\{a_n\}$ is either increasing or decreasing we say a_n is *monotonic.*

4. If there exists a number m such that $m \leq a_n$ for every n we say $\{a_n\}$ is *bounded below* by m.

5. If there is a number M such that $a_n \leq M$ for every n we say the sequence is *bounded above.*

6. If the sequence is bounded above and below, we say the sequence is *bounded.*

Example 11. Determine if the following sequences are monotonic and/or bounded.

(a) $\{-n^2\}$

(b) $(-1)^{n+1}$

(c) $\{2/n^2\}$

Solution. (a) is monotonic, but not bounded as for any $M < 0$, there is eventually an integer N such that $-N < M$.

(b) is bounded but not monotonic. It is not monotonic because the first three terms of the sequence are $-1, 1, -1$ so $a_1 < a_2$, but $a_2 > a_1$. It is bounded because the sequence never goes above 1 or below $-1>$

(c) The sequence is both monotonic and bounded since

$$a_n = \frac{2}{n^2} < \frac{2}{(n-1)^2} = a_{n-1}$$

as

$$\begin{aligned} 1 &> \frac{(n-1)^2}{n^2} \\ &= \left(\frac{n-1}{n}\right)^2 \\ &= \left(1 - \frac{1}{n}\right)^2. \end{aligned}$$

It is bounded since the largest term of the sequence is 2 (upper bound) and the sequence approaches, but never reaches, 0 (the lower bound, and the limit). ◇

Example 12. Determine if the following sequences are monotonic and/or bounded.

(a) $\{n/(n+1)\}$

(b) $\{n^3\}n^4 + 10000$

Solution. ◇

Theorem 13. *If $\{a_n\}$ is a bounded and monotonic sequence, then $\{a_n\}$ converges.*

Remark. This theorem applies the very first example we looked at, (1). Note that for this sequence,

$$a_n = \frac{n}{n+1} > \frac{n-1}{n} = a_{n-1}$$

since, after rearranging the inequality above,

$$n^2 > (n+1)(n-1) = n^2 - 1,$$

so the sequence is monotonic and increasing (often, we say monotonically increasing). Moreover, it is bounded since

$$a_n = \frac{n}{n+1} < 1$$

so

$$n < n+1,$$

which is true. Theorem 13 would then tell us that the sequence must converge, although it does not tell us what it converges to.

Remark. You may have noticed that oftentimes, the limit of a sequence can be used as a bound. This is an excellent observation and this can help you find the limit of a sequence. That is, finding the optimal upper (or lower bound, as it may be) of a bounded sequence can help you determine its limit