MA 162 Lecture 9
July 11, 2019

Lecture

Today we will be introducing two more results—the ratio test, and the root test—which will let us determine the convergence of series with slightly more complicated terms.

The ratio test

The ratio test is particularly useful for determining the convergence of series with factorial terms. Let us recall what a factorial is. Let n be a counting number. Then $n!$ is defined (recursively) as

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n(n-1)! & \text{otherwise.} \end{cases}$$

Example 1. Here are some sample computations of the factorial.

$$1! = 1 \cdot 0! = 1 \cdot 1 = 1.$$

$$2! = 2 \cdot 1! = 2 \cdot 1 = 2.$$

$$3! = 3 \cdot 2! = 3 \cdot 2 = 6.$$

$$4! = 4 \cdot 3! = 4 \cdot 6 = 24.$$

Now we are ready to introduce the ratio test.

Theorem (Ratio test)**.** *Suppose* $\sum_{n=k}^{\infty} a_n$ *is a series. Define*

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then

(1) *if* $L < 1$ *the series is absolutely convergent;*

(2) *if* $L > 1$ *the series is divergent;*

(3) *if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.*

Example 2. Determine if the series

$$\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$$

is convergent or divergent.

Solution. First, write the ratio

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-10)^{n+1}}{4^{2n+3}(n+2)} \Big/ \frac{(-10)^n}{4^{2n+1}(n+1)}\right| \\ &= \left|\left(\frac{(-10)^{n+1}}{4^{2n+3}}\right)\left(\frac{4^{2n+1}(n+1)}{(-10)^n}\right)\right| \\ &= \left|\frac{-10(n+1)}{(4^2(n+2))}\right|.\end{aligned}$$

Taking the limit of this, and using L'Hôpital's rule, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{10}{16} = \frac{5}{8} < 1.$$

Therefore, by (1) of the ratio test, the series converges. ◇

Example 3. Determine if the series

$$\sum_{n=0}^{\infty} \frac{n!}{5^n}$$

is convergent or divergent.

Solution. Write the ratio

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\left(\frac{(n+1)!}{5^{n+1}}\right)\left(\frac{5^n}{n!}\right)\right| \\ &= \frac{(n+1)!}{5n!} \\ &= \frac{(n+1)}{5}.\end{aligned}$$

Evidently

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)}{5} = \infty.$$

Thus, by the ratio test, the series diverges. ◇

Example 4. Determine if the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$$

is convergent or divergent.

Solution. Write the ratio and take the limit:

$$\lim_{n\to\infty}\left|\left(\frac{(-1)^{n+1}}{(n+1)^2+1}\right)\left(\frac{n^2+1}{(-1)^n}\right)\right| = \lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1} = 1.$$

The ratio terns turns up inconclusive. However, since

$$\lim_{n\to\infty}\frac{1}{n^2+1} = 0,$$

by the alternating series test, this series converges. ◇

The root test

The root test is a bit more situational so we saved it for last.

Theorem (Root test)**.**

Let $\sum_{n=k}^{\infty} a_n$ be a series. Define

$$L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}|a_n|^{1/n}.$$

Then

(1) if $L < 1$ the series is absolutely convergent;

(2) if $L > 1$ the series diverges;

(3) if $L = 1$ the test is inconclusive.

Example 5. Determine if the series

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{2n+1}}$$

is convergent or divergent.

Solution. By the root test

$$\lim_{n\to\infty} \left| \frac{n^n}{3^{1+2n}} \right|^{1/n} \lim_{n\to\infty} \frac{n}{3^{1/n+2}} = \infty > 1.$$

Therefore, by the root test, the series diverges. ◇

Example 6. Determine if the series

$$\sum_{n=0}^{\infty} \left(\frac{5n - 3n^3}{7n^3 + 2} \right)^n$$

is convergent or divergent.

Solution. Write down the n-th root of the series and take the limit

$$\lim_{n\to\infty} \left| \left(\frac{5n - 3n^3}{7n^3 + 2} \right)^n \right|^{1/n} \lim_{n\to\infty} \left| \frac{5n - 3n^3}{7n^3 + 2} \right| = \frac{3}{7} < 1.$$

Therefore, by the root test, the series converges. ◇