# MA 162 Quiz 8
## July 18, 2019

You have **15 minutes** to complete this quiz. Each correct answer will award you **five points**. Show your work **neatly** and you will receive **one** to **three points** depending on your level of correctness.

**Problem 8.1.** Which of the following gives the third order Taylor polynomial for $f(x) = \sin x$ about $x = \pi/2$?

(A) $1 - \frac{1}{2!}(x - \pi/2)^2$
(B) $-(x - \pi/2) + \frac{1}{3!}(x - \pi/2)^3$
(C) $-(x - \pi/2) - \frac{1}{3!}(x - \pi/2)^3$
(D) $-(x - \pi/2) + (x - \pi/2)^3$
(E) $1 + \frac{1}{2!}(x - \pi/2)^2$

*Solution.* By the formula for the $n^{\text{th}}$ Taylor polynomial centered at $a$, i.e.

$$p_n(x) = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k.$$

With $n = 3$ and $a = \pi/2$, we have

$$\begin{aligned} f(\pi/2) &= \sin(\pi/2) = 1, \\ f'(\pi/2) &= \cos(\pi/2) = 0, \\ f''(\pi/2) &= -\sin(\pi/2) = -1, \\ f'''(\pi/2) &= -\cos(\pi/2) = 0, \end{aligned}$$

so

$$p_3(x) = 1 - \tfrac{1}{2!}(x - \pi/2)^2.$$

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**Problem 8.2.** Determine for which $x$ is the approximation of $\sin x$ by its fourth order Taylor polynomial about $x = 0$ accurate within $1/3\,840$ by using the remainder theorem. *Hint:* $3\,840 = 5! \cdot 2^5$.

(A) $-1/2 < x < 1/2$
(B) $-\sqrt[5]{120} < x < \sqrt[5]{120}$
(C) $-1 < x < 1$

(D) $-120 < x < 120$
(E) $-\sqrt{3}2 < x < \sqrt{3}2$.

*Solution.* From Taylor's remainder theorem,

$$|R_n(c)| \leq M\frac{|c-a|^{n+1}}{(n+1)!},$$

where $M \geq f^{(n+1)}(c)$ for all $c$ in a neighborhood of $a$.

In this case, $n = 4$ and $a = 0$. Moreover,

$$\frac{d^n}{dx^n}[\sin x] = \begin{cases} |\sin x| & \text{if } n \text{ is even,} \\ |\cos x| & \text{if } n \text{ is odd,} \end{cases} \leq 1.$$

So setting $M = 1$,

$$|R_n(c)| \leq \frac{|c|^5}{5!}.$$

Now, we want the error to remain within $1/3\,840$ so using the above estimate

$$\frac{|c|^5}{5!} \leq \frac{1}{3\,840} = \frac{1}{5!2^5},$$

i.e. $|c| \leq 1/2$. So $c$ can take any values between $-1/2$ and $1/2$. Since $c$ here is just a variable, we can replace it by $x$ to get the answer. ◇

**Problem 8.3.** The following is the fourth order Taylor polynomial of a function $f(x)$ at $a$:

$$T_4(x) = 10 + 5(x-a) + \sqrt{3}(x-a)^2 + \tfrac{1}{2\pi}(x-a)^3 + 17e(x-a)^4.$$

What is $f'''(a)$?

(A) $2\sqrt{3}$
(B) $1/2\pi$
(C) $17e$
(D) $1/6\pi$
(E) $3/\pi$

*Solution.* Recall, by the formula for the $n^{\text{th}}$ order Taylor polynomial, the coefficient in front of the $(x-a)^k$-term is $f^{(k)}/k!$ so if we want the fourth derivative of $f$ at $a$ we need

$$\left(\frac{f^{(4)}(a)}{4!}\right)4! = \left(\frac{1}{2\pi}\right)6! = \frac{3}{\pi}.$$

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