MA 261 Exam 2 Solutions

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Problem 2.1. The extreme values of $f(x, y, z) = 3x+2y+6z$ with constraint $x^2 + y^2 + z^2 = 4$ are

Solution. By the method of Lagrange multipliers, we must find values of x, y, z and λ satisfying $\operatorname{grad} f = \lambda \operatorname{grad} g$, where $g = x^2 + y^2 + z^2 - 4$, i.e.,

$$\begin{aligned} 3 &= 2\lambda x, \\ 2 &= 2\lambda y, \\ 6 &= 2\lambda z. \end{aligned} \tag{2.1}$$

From Equations (2.1) we can see that $y = \frac{2}{3}x = \frac{1}{3}z$, or $z = 3y$, $x = \frac{3}{2}y$. Plugging these into the constraint

$$\begin{aligned} 4 &= \frac{9^{\,2}}{y} + y^2 + 9y^2 \\ &= \left(\frac{9}{4} + 1 + 9\right)y^2 \\ &= \frac{49}{4}y^2, \end{aligned}$$

so $y = \pm 4/7$. Taking this value of y and putting it into the relations we obtained from Equations (2.1), we get $x = \pm 6/7$ and $z = \pm 12/7$. Therefore,

$$\begin{aligned} f(\pm 6/7, \pm 4/7, \pm 12/7) &= \pm(18/7 + 8/7 + 72/7) \\ &= \pm\frac{98}{7} \\ &= \pm 14. \end{aligned}$$

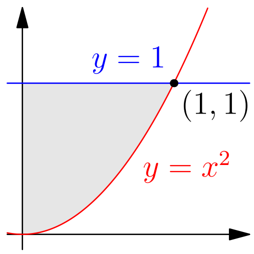
Therefore, the maximum must be 14 and minimum -14, subject to the constraint.

Answers: (B), (D). ◇

Problem 2.2. Reverse the order of integration and evaluate the integral

$$\int_0^1 \int_{x^2}^1 6\sqrt{y}\cos(y^2)\,dydx.$$

Solution. After sketching the region of integration, as we do below,



we see that the integral can be easily rewritten as

$$\int_0^1 \int_0^{\sqrt{y}} 6\sqrt{y}\cos(y^2)\,dxdy,$$

and this we can easily compute as we do below:

$$\int_0^1 \int_0^{\sqrt{y}} 6\sqrt{y}\cos(y^2)\,dxdy = \int_0^1 6y\cos(y^2)\,dy,$$

making the u-substitution, $u = y^2$, $du = 2y\,dy$, this simplifies into

$$\begin{aligned} &= \int_0^1 3\cos(u)\,du \\ &= 3\sin(u)|_0^1 \\ &= 3\sin 1 - 3\sin 0 \\ &= 3\sin 1. \end{aligned}$$

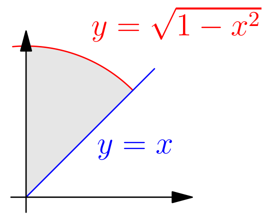
Answers: (C), (E). ◇

Problem 2.3. Evaluate

$$\int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} 3\sqrt{x^2+y^2}\,dydx$$

using polar coordinates.

Solution. If we sketch the region being traced out by the bounds in the double integral, as we do below



we see that, in polar coordinates, it is a sector with $0 \leq r \leq 1$, and $\pi/4 \leq \theta \leq \pi/2$

In polar coordinates, our integral will take the form

$$\int_{\pi/4}^{\pi/2} \int_0^1 3r^2 \, dr d\theta.$$

This we can easily compute:

$$\begin{aligned} \int_{\pi/4}^{\pi/2} \int_0^1 3r^2 \, dr d\theta &= \int_{\pi/4}^{\pi/2} d\theta \\ &= \frac{\pi}{4}. \end{aligned}$$

Answers: (D), (C). ◇

Problem 2.4. Find the area of the part of the plane $3x + 2y + z = 6$ that is in the first octant.

Solution. Parameterize the plane by $\mathbf{r}(u, v) = \langle u, v, 6 - 3u - 2v \rangle$. Then,

$$\begin{aligned} \mathbf{r}_u(u, v) &= \langle 1, 0, -3 \rangle, \\ \mathbf{r}_v(u, v) &= \langle 0, 1, -2 \rangle, \\ \mathbf{r}_u \times \mathbf{r}_v(u, v) &= \langle 3, 2, 1 \rangle. \end{aligned}$$

Moreover, the part of the part of the plane which lies in the first octant requires that $x, y, z \geq 0$, so when $z = 0$, $0 \leq u \leq 2$ and $0 \geq v \leq 3 - \frac{3}{2}u$

$$\begin{aligned} \int_0^2 \int_0^{3-\frac{3}{2}u} \sqrt{14} \, dv du &= \sqrt{14} \int_0^2 \left(3 - \frac{3}{2}u\right) du \\ &= \sqrt{14} \Big[3u - 3/4u^2\Big]_0^2 \\ &= 3\sqrt{14}. \end{aligned}$$

Answers: (B), (D). ◇

Problem 2.5. Consider the tetrahedron E with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$. Express

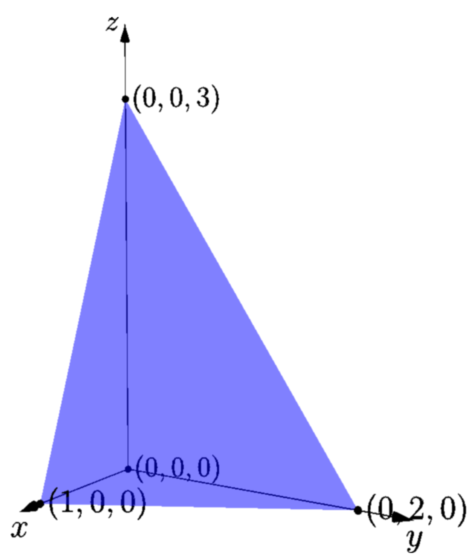
$$\iiint_E x\, dV$$

as an iterated integral in the order $dzdydx$.

Solution. Since we are writing the integral in the order $dzdydx$ we will, for now, q ignore the origin $(0,0,0)$ because it is on the same axis as each of the other points. Now, the first order of business is to determine the plane cutting through each of the other points. To this end, write

$$\begin{aligned}(1,0,0)-(0,0,3) &= \langle 1,0,-3\rangle,\\ (0,2,0)-(0,0,3) &= \langle 0,2,-3\rangle,\\ \langle 1,0,-3\rangle \times \langle 0,2,-3\rangle &= \langle 6,3,2\rangle.\end{aligned}$$

Therefore, this plane is of the form $6x+3y+2z=d$ with $d=6$ since $(1,0,0)$ is a point in this plane, so $6x+3y+2z=6$. The relevant segment of this plane is sketched below



Now, if we were to sketch the region, we would see that $0 \le z \le -3x - \frac{3}{2}y + 3$, $0 \le y \le 2-2x$, and $0 \le x \le 1$. So the correct integral must be

$$\int_0^1 \int_0^{2-2x} \int_0^{-3x-\frac{3}{2}y+3} x\, dzdydx.$$

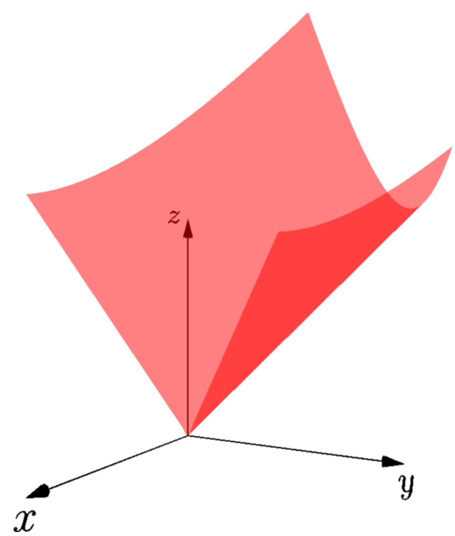
Answers: (C), (A). ◇

Problem 2.6. The triple integral

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{x^2+y^2}} 8(x^2+y^2)\,dzdydx,$$

when converted to cylindrical coordinates becomes

Solution. The solid region described, has the following graph



This is the segment of a cone cut by the x axis and reaching the value $r^2 = 9$. Therefore, in cylindrical coordinates, we have

$$0 \le r \le 3, \quad 0 \le \theta \le \pi, 0 \le z \le r,$$

so

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{x^2+y^2}} 8(x^2+y^2)\,dzdydx = \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{r} 8r^3\,dzdrd\theta.$$

Answers: (A), (D). ◇

Problem 2.7. Evaluate the triple integral $\iiint_E (x^2+y^2)\,dV$ where E is the solid region in the first octant which is outside the sphere $x^2+y^2+z^2=1$ and inside $x^2+y^2+z^2=4$.

Solution. This problem is best approached by changing to spherical coordinates. If we make this change, the integral is easily computed:

$$\begin{aligned}
\iint_E x^2 + y^2\, dV &= \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^2 \sin^2 \phi (\rho^2 \sin \phi)\, d\rho d\phi d\theta \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^4 \sin^3 \phi\, d\rho d\phi d\theta \\
&= \left(\int_0^{\pi/2} d\theta \right) \left(\int_0^{\pi/2} \sin^3 \phi\, d\phi \right) \left(\int_1^2 \rho^4\, d\rho \right) \\
&= \left(\frac{\pi}{2} - 0 \right) \left(-\cos(\pi/2) + \tfrac{1}{3}\cos^3(\pi/2) + \cos 0 - \tfrac{1}{3}\cos^3 0 \right) \left(\frac{2^5}{5} - \frac{1}{5} \right) \\
&= \frac{\pi}{2} \cdot \frac{2}{3} \cdot \frac{31}{5} \\
&= \frac{31\pi}{15}
\end{aligned}$$

To compute the integral $\int_0^{\pi/2} \sin^3 \phi\, d\phi$, use the Pythagorean theorem to turn $\sin^3 \phi$ into $\sin \phi (1 - \cos^2 \phi) = \sin \phi - \sin \phi \cos^2 \phi$ and then use u substitution with $u = \cos \phi$ to arrive at

$$\int \sin^3 \phi\, d\phi = -\cos \phi + \tfrac{1}{3} \cos^3 \phi + C.$$

Answers: (E), (A). ◇

Problem 2.8. Let $f(x, y, z) = x^2 + y^3 + z^4$ and $g(x, y, z) = 3x + 4y + z^2/2$. If $\nabla f(2, 1, -1)$ is perpendicular to $\nabla g(a, b, c)$, then

Solution. Recall that two vector $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. Therefore, we must find a point (a, b, c) such that the vectors $\operatorname{grad} f(2, 1, -1)$ and $\operatorname{grad} g(a, b, c)$ are perpendicular. But first we need to find what $\operatorname{grad} f$ and $\operatorname{grad} g$ are:

$$\begin{aligned}
\operatorname{grad} f(x, y, z) &= \langle 2x, 3y^2, 4z^3 \rangle, \\
\operatorname{grad} g(x, y, z) &= \langle 3, 4, z \rangle.
\end{aligned}$$

Thus, $\operatorname{grad} f(2, 1, -1) = \langle 4, 3, -4 \rangle$ and $\operatorname{grad} g(a, b, c) = \langle 3, 5, c \rangle$ and for these vectors to be perpendicular, we must have

$$\operatorname{grad} f(2, 1, -1) \cdot \operatorname{grad}(a, b, c) = 12 + 12 - 4c = 0$$

so $c = 24/4 = 6$.
Answers: (C), (D). ◇

Problem 2.9. Evaluate the line integral $\int_C xy\,dx - y^2\,dy$, where C is the line segment from $(0,0)$ to $(2,6)$.

Solution. The first thing we must do is parametrize the line segment from $(0,0)$ to $(2,6)$. This can always be done in the same way, i.e., for a point P and Q, the line segment from P to Q is $\mathbf{r}(t) = Qt + (1-t)P$, so in our case it is

$$\mathbf{r}(t) = (2,6)t + (1-t)(0,0) = \langle 2t, 6t\rangle, \quad 0 \le t \le 1.$$

To compute the line integral, we will need $\mathbf{r}'(t)$, which is

$$\mathbf{r}'(t) = \langle 2, 6\rangle.$$

Putting all of this information together, we can calculate the line integral as follows

$$\begin{aligned}
\int_C xy\,dx - y^2\,dy &= \int_0^1 2(2t)(6t) - 6(6t)^2\,dt \\
&= \int_0^1 2(2t)(6t) - 6(6t)^2\,dt \\
&= \int_0^1 (2\cdot 2\cdot 6 - 6\cdot 6\cdot 6)t^2\,dt \\
&= \int_0^1 -32\cdot 6t^2\,dt \\
&= -32\cdot 6\left[\frac{t^3}{3}\right]_0^1 \\
&= -64.
\end{aligned}$$

Answers: (D), (B). ◇

Problem 2.10. Evaluate the line integral $\int_C 9x/y\,ds$, where C is the curve $x = t^3/3$, $y = t^4/4$ with $1 \le t \le 2$.

Solution. To compute the line integral, we need to find $abs\mathbf{r}'(t)$. The curve is $\mathbf{r}(t) = \langle t^3/3, t^4/4\rangle$ and its derivative is $\mathbf{r}'(t) = \langle t^2, t^3\rangle$ so

$$|\mathbf{r}'(t)| = \sqrt{t^4 + t^6} = t^2\sqrt{1+t^2}.$$

Therefore, the line integral is

$$\begin{aligned}
\int_C \frac{9x}{y}\,ds &= \int_1^2 \frac{12}{t} t^2\sqrt{1+t^2}\,dt \\
&= 12\int_1^2 t\sqrt{1+t^2}\,dt
\end{aligned}$$

which, by u substitution with $u = 1 + t^2$, becomes

$$
\begin{aligned}
&= 12 \int u^{1/2} \, \frac{du}{2} \\
&= 6 \int_2^5 u^{1/2} \, du \\
&= 6 \left[\frac{2}{3} u^{3/2} \right]_2^5 \\
&= 4(5^{3/2} - 2^{3/2}).
\end{aligned}
$$

Answers: (B), (E). $\diamond$