MA 261 Quiz 12
April 16, 2019

If you do not know how to do any one of these problems, circle **"(E) I don't know"** as your answer choice. You will receive **one point** for doing that. **Each problem** is worth **five points**. You get **two points** for writing your **full name** and **three points** for writing your **PUID and section number**.

Problem 12.1. Let S be the surface parametrized by $\mathbf{r}(u, v) = \langle v \cos u, v \sin u, 2v^2 \rangle$ with $0 \leq u \leq 2$ and $0 \leq v \leq 2$. Then S is part of a

(A) circular paraboloid
(B) cone
(C) cylinder
(D) ellipsoid
(E) I don't know how to do this problem

Solution. Write $x = v \cos u$, $y = v \sin u$, and $z = 2v^2$. Then, note that $x^2 + y^2 = z$. This is a paraboloid with circular cross sections, so it is a circular paraboloid.
Answer: (A). ◇

Problem 12.2. Let S be the part of the sphere $x^2 + y^2 + z^2 = 1$ above the plane $z = 1/2$. Evaluate the surface integral

$$\iint_S 12z^2 \, dS.$$

Hint: Use the parametrization $\mathbf{r}(u, v) = \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle$ and apply u-substitution.

(A) 2π
(B) π
(C) 7π
(D) 8π
(E) I don't know how to do this problem

Solution. First, we need to find a parametrization of the surface. Since the surface is part of a sphere, there is a natural parametrization and it is .

$$\mathbf{r}(\theta, \phi) = \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle.$$

Its partial derivatives are

$$\mathbf{r}_\theta(\theta,\phi) = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle,$$
$$\mathbf{r}_\phi(\theta,\phi) = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle.$$

Next, we need to find $\mathbf{r}_\theta \times \mathbf{r}_\phi$. This computation can get a little hairy, but nevertheless with perseverance, we get

$$\begin{aligned}\mathbf{r}_\theta \times \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{vmatrix} \\ &= \langle -\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi\sin^2\theta - \sin\phi\cos\phi\cos^2\theta\rangle \\ &= \langle -\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi\rangle\end{aligned}$$

So the norm is

$$\begin{aligned}|\mathbf{r}_\theta \times \mathbf{r}_\phi| &= \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi} \\ &= \sqrt{\sin^4\phi(\cos^2\theta + \sin^2\theta) + \sin^2\phi\cos^2\phi} \\ &= \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} \\ &= \sqrt{\sin^2\phi(\sin^2\phi + \cos^2\phi)} \\ &= \sqrt{\sin^2\phi} \\ &= \sin\phi.\end{aligned}$$

Therefore, the surface integral is

$$\iint_S 12z^2\, dS = \int_0^{2\pi}\int_0^{\pi/3} 12\cos^2\phi\sin\phi\, d\phi d\theta$$

making the u-substitution, $u = \cos\phi$

$$\begin{aligned}&= \int_0^{2\pi}\int_1^{1/2} -12u^2\, du d\theta \\ &= 2\pi\int_{1/2}^1 12u^2\, du \\ &= 8\pi\Big[u^3\Big]_{1/2}^1 \\ &= 8\pi(1 - 1/8) \\ &= 7\pi.\end{aligned}$$

Answer: (C). ◇