MA 527 Section 1
Solutions to Selected Problems from Problem Set 2

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Graded problems:

Section	Number
7.3	6, 7, 13
7.4	7, 14, 33

Problem 2.1 (7.3 # 6)**.** Solve the linear system given explicitly or by its augmented matrix. Show details.

$$\begin{pmatrix} 4 & -8 & 3 & 16 \\ -1 & 2 & -5 & -21 \\ 3 & -6 & 1 & 7 \end{pmatrix}$$

Solution. To solve the linear systems in problems 6, 7, and 13 from Section 7.3, we will use Gaussian elimination. Unfortunately, in the interest of time, we will not provide detailed solutions. By Gaussian elimination, we have

$$\begin{pmatrix} 4 & -8 & 3 & 16 \\ -1 & 2 & -5 & -21 \\ 3 & -6 & 1 & 7 \end{pmatrix} \implies \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

At a glance, that is, by converting the matrix into the equivalent linear system below

$$\begin{aligned} x - 2y &= 1 \\ z &= 4, \end{aligned}$$

we can see that the system has a solution set comprised of vectors of the form

$$\begin{pmatrix} 1+2y \\ y \\ 4 \end{pmatrix}$$

for any real number y. $\diamond$

Problem 2.2 (7.3 # 7).

$$\begin{pmatrix} 2 & 4 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ 4 & 0 & 6 & 0 \end{pmatrix}$$

Solution. By Gaussian elimination, we have

$$\begin{pmatrix} 2 & 4 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ 4 & 0 & 6 & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is difficult to read the solution from this, but if we write out the linear system, we have

$$\begin{aligned} x - y + 2z &= 0 \\ 2y - z &= 0 \end{aligned}$$

so $z = 2y$, and $x - y + 2z = x - y + 4y = 0 \implies x = -3y$. Therefore, the solution set consists of vectors of the form

$$\begin{pmatrix} -3y \\ y \\ 2y \end{pmatrix}$$

for any real y. $\diamond$

Problem 2.3 (7.3 # 13).

$$\begin{aligned} 10x + 4y - 2z &= -4 \\ -3w - 17x + y + 2z &= 2 \\ w + x + y &= 6 \\ 8w - 34x + 16y - 10z &= 4. \end{aligned}$$

Solution. We can rewrite the linear system above as the augmented system

$$\begin{pmatrix} 0 & 10 & 4 & -2 & -4 \\ -3 & -17 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 6 \\ 8 & -34 & 16 & -10 & 4 \end{pmatrix}$$

and do elimination until we get a diagonal of 1 staring at the top left corner, like so

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}.$$

Then, the solution is the rightmost column of the matrix above, i.e.

$$\mathbf{x} = \begin{pmatrix} 4 \\ 0 \\ 2 \\ 6 \end{pmatrix}.$$

◇

Problem 2.4 (7.4 # 7)**.** Find the rank. Find a basis for the row space. Find a basis for the column space.

$$\begin{pmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 4 & 0 & 2 & 0 \end{pmatrix}.$$

Solution. By performing Gaussian elimination on the matrix, we get

$$\begin{pmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 4 & 0 & 2 & 0 \end{pmatrix} \implies \begin{pmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the vectors $(8,0,4,0)$ and $(0,2,0,4)$ are clearly independent. Therefore, $\operatorname{rk} A = 2$ and a basis for the row space of the matrix could be $\{(8,0,4,0),(0,2,0,4)\}$.

Similarly, $\operatorname{rk} A = \operatorname{rk}(A^{\mathrm{T}}) = 2$ and, by elimination, we have

$$\begin{pmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 4 & 0 & 2 & 0 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 2 \\ 0 & 4 & 0 \end{pmatrix} \implies \begin{pmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the column space could be $\{(8,0,4),(0,2,0)\}$. ◇

Problem 2.5 (7.3 # 14)**.** If $\mathbf{A}$ is not square, either the row vectors or the column vectors of $\mathbf{A}$ are linearly independent.

Solution. A proof by (a) contradiction quoting Theorem 3 (implicitly or explicitly), or (b) a direct proof quoting Theorem 4 from Section 7.4 suffices here (with the right buzzwords of course).

(a) Let $\mathbf{A}$ be an $m \times n$ matrix with $m \neq n$. By contradiction, we will assume that both the rows and the columns of $\mathbf{A}$ are linearly independent. Since the rows of A are independent, $\operatorname{rk} A = n$. However, $\operatorname{rk} A = \operatorname{rk}(A^{\mathrm{T}})$ by Theorem 3, so $\operatorname{rk}(A^{\mathrm{T}}) = n$. However, by definition $\operatorname{rk}(A^{\mathrm{T}})$ is the number of linearly independent columns which, by the assumption we have made, is m. Therefore, $m = n$. But this is clearly nonsensical because $m = n$ and $m \neq n$ cannot both be true. Therefore, our assumption that both the rows and the columns of $\mathbf{A}$ are linearly independent must be false; i.e., either the rows or the columns are dependent. (If you do not understand this argument, review De Morgan's laws.)

(b) Let $\mathbf{A}$ be an $m \times n$ matrix with $m \neq n$. Without loss of generality, we may assume $m > n$. (Why? We are working with a nonsquare matrix so $m \neq n$. If you give me a matrix with $m < n$, I can take the transpose of that matrix and now I have an $n \times m$ matrix which has more rows than columns.) Let $\mathbf{r}_1, \ldots, \mathbf{r}_m$ denote the column vectors of $\mathbf{A}$. Then, we have m vectors having n components, so by Theorem 4, these vectors must be linearly dependent because $m > n$. $\diamond$

Problem 2.6 (7.3 # 33)**.** Is the given set of vectors a vector space? Give reasons. If your answer is yes, determine the dimension and find a basis. ($v_1, v_2, \ldots$ denote components.)

All vectors in $\mathbb{R}^3$ with $3v_1 - v_3 = 0$, $2v_1 + 3v_2 - 4v_3 = 0$.

Solution. Since the vectors live in $\mathbb{R}^3$, the set of solutions to the linear system, if it is a vector space, will be a subspace of $\mathbb{R}^3$. So we need to check two things: (1) that the zero vector $\mathbf{0} = (0, 0, 0)$ is a solution and (2) that given any two solutions $\mathbf{u}, \mathbf{w}$, their sum $\mathbf{u} + \mathbf{w}$ is a solution.

(1) is obvious (just plug in the zero vector into the system and notice that the equations are satisfied).

For (2), notice that

$$\begin{aligned}
3(u_1 + w_1) - (u_3 + w_3) &= (3u_1 - u_3) + (3w_1 - w_3) \\
&= 0 + 0 = 0. \\
2(u_1 + w_1) + 3(u_2 + w_2) - 4(u_3 + w_3) &= (2u_1 + 3u_2 - 4u_3) + (2w_1 + 3w_2 - 4w_3) \\
&= 0 + 0 = 0,
\end{aligned}$$

thus $\mathbf{u} + \mathbf{v}$ is a solution.

A possible basis for this subspace is $\{(1, 10/3, 3)\}$. ◇