MA 527 Section 1
Solutions to Selected Problems from Problem Set 3

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Graded problems:

Section	Number
7.4	15, 26
7.7	14, 18, 24

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Problem 3.1 (7.4 # 15)**.** If the row vectors of a square matrix are linearly independent, so are the column vectors, and conversely.

Solution. Let A be an $n \times n$ matrix. Then, $\operatorname{rank} A = n$ if and only if the rows of A are linearly independent. Since $\operatorname{rank} A = \operatorname{rank} A^{\mathrm{T}}$, by Theorem 6, the rows of A are linearly independent if and only if the columns of A are linearly independent. $\diamond$

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Problem 3.2 (7.4 # 26)**. Linearly independent subset.** Beginning with the last of the vectors $(3, 0, 1, 2)$, $(6, 1, 0, 0)$, $(12, 1, 2, 4)$, $(6, 0, 2, 4)$, and $(9, 0, 1, 2)$, omit one after another until you get a linearly independent set.

Solution. One thing which will serve us well for this problem is to write down the set of vectors as the rows of the matrix $\mathbf{A}$, as we do below

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 6 & 1 & 0 & 0 \\ 12 & 1 & 2 & 4 \\ 6 & 0 & 2 & 4 \\ 9 & 0 & 1 & 2 \end{bmatrix}$$

and perform Gaussian elimination on $\mathbf{A}$ being careful to keep track of which vector used to be which.

Without further ado, by elimination we have

$$\mathbf{A} \implies \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so there are at most three linearly independent vectors. The vectors are $(3, 0, 1, 2)$, $(6, 1, 0, 0)$, and $(9, 0, 1, 2)$ since

$$\begin{aligned} (1, 0, 0, 0) &= \tfrac{1}{6}\Big[(9, 0, 1, 2) - (3, 0, 1, 2)\Big] \\ (0, 1, 0, 0) &= (6, 1, 0, 0) + (3, 0, 1, 2) - (9, 0, 1, 2) \\ (0, 0, 1, 2) &= -\tfrac{1}{2}\Big[(9, 0, 1, 2) - 3(3, 0, 1, 2)\Big]. \end{aligned}$$

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Problem 3.3 (7.7 # 14)**.** Showing the details, evaluate:

$$\begin{vmatrix} 4 & 7 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -2 & 2 \end{vmatrix}.$$

Solution. We will mostly be using (b) from Theorem 1 to compute these determinants. Recall what that says: *addition of a multiple of a row to another row does not change the value of the determinant.* Therefore,

$$\begin{vmatrix} 4 & 7 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 0 & 0 \\ 0 & \frac{9}{2} & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 12 \end{vmatrix} = 4 \cdot \tfrac{9}{2} \cdot 1 \cdot 12 = 216.$$

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Problem 3.4 (7.7 # 18)**.** Find the rank by Theorem 3 (which is not very practical) and check by row reduction. Show details

$$\begin{bmatrix} 0 & 4 & -6 \\ 4 & 0 & 10 \\ -6 & 10 & 0 \end{bmatrix}.$$

Solution. Recall that Theorem 3 that a matrix $\mathbf{A}$ has rank $r \geq 1$ if and only if it contains an $r \times r$ submatrix whose determinant is nonzero. Now,

$$\begin{vmatrix} 0 & 4 & -6 \\ 4 & 0 & 10 \\ -6 & 10 & 0 \end{vmatrix} = -\begin{vmatrix} 4 & 0 & 10 \\ 0 & 4 & -6 \\ 0 & 0 & 30 \end{vmatrix} = -480 \neq 0.$$

Therefore, the rank of the matrix must be 3. Note that in the process of finding the determinant we performed elimination on the original matrix, so we need not do this again. ◇

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Problem 3.5 (7.7. # 24)**.** Solve by Cramer's rule. Check by Gauss elimination and back substitution. Show details.

$$\begin{aligned} 3x - 2y + z &= 13 \\ -2x + y + 4z &= 11 \\ x + 4y - 5z &= -31. \end{aligned}$$

Solution. By Cramer's rule, we have

$$\begin{aligned} D &= \begin{vmatrix} 3 & -2 & 1 \\ -2 & 1 & 4 \\ 1 & 4 & -5 \end{vmatrix} = -60, \\ x &= \frac{\begin{vmatrix} 13 & -2 & 1 \\ 11 & 1 & 4 \\ -31 & 4 & -5 \end{vmatrix}}{-60} = \frac{-60}{-60} = 1, \\ y &= \frac{\begin{vmatrix} 3 & 13 & 1 \\ -2 & 11 & 4 \\ 1 & -31 & -5 \end{vmatrix}}{-60} = -3, \\ z &= \frac{\begin{vmatrix} 3 & -2 & 13 \\ -2 & 1 & 11 \\ 1 & 4 & -32 \end{vmatrix}}{-60} = 4. \end{aligned}$$

By back substitution,

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\begin{aligned}
3(1) - 2(-3) + 4 &= 3 + 6 + 4 = 13, \\
-2(1) + (-3) + 4(4) &= -2 - 3 + 16 = 11, \\
1 + 4(-3) - 5(4) &= 1 - 12 - 20 = -31.
\end{aligned}
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