Topic: Continuous Annuities

Ram is receiving a continuous annuity with payments of $13t^2$ at time t for 16 years.

Using a discount function of $v(t) = 1 - 0.03t$, calculate the present value of Ram's annuity.

Solution:

$$PV = \int_0^{16} (13t^2)(1 - 0.03t)dt$$

$$= \int_0^{16} (13t^2 - 0.39t^3)dt$$

$$= \left[\frac{13}{3}t^3 - \frac{0.39}{4}t^4 \right]_0^{16}$$

$$= 11{,}359.57$$

Claire is receiving a 25 year continuous annuity with payments of $1234t$ at time t.

Using a force of interest of 6%, calculate the present value of Claire's annuity.

Solution:

$$\int_0^{25} (1234t)v^t\,dt = 1234\left(\bar{I}\bar{a}\right)_{\overline{25|}}$$

$$= 1234\left(\frac{\dfrac{1-e^{-(25)(0.06)}}{0.06} - 25e^{-(25)(0.06)}}{0.06}\right)$$

$$= 1234(122.83)$$

$$= 151{,}567.63$$

Dylan is receiving a continuous annuity that pays at a rate of $1000t+500$ at time t for 25 years.

Calculate the present value at $\delta = 0.09$.

Solution:

$$\int_0^{25}(1000t+500)v^t\,dt = \int_0^{25}(1000t)v^t\,dt + \int_0^{25}(500)v^t\,dt = 1000\left(\bar{I}\bar{a}\right)_{\overline{25}|} + 500\bar{a}_{\overline{25}|}$$

$$= 1000\left(\frac{\dfrac{1-e^{-(25)(0.09)}}{0.09} - 25e^{-(25)(0.09)}}{0.09}\right) + 500\left(\frac{1-e^{-(25)(0.09)}}{0.09}\right)$$

$$= 81{,}166.98 + 4970.00 = 86{,}136.98$$

Ian can purchase any of the following perpetuities. All three perpetuities have the same price at an annual interest rate of i.

a. Perpetuity A pays continuously at a rate of 20,000 per year.

b. Perpetuity B pays continuously at a rate of 1000t at time t.

c. Perpetuity C has quarterly payments at the end of each quarter. Each payment in the first year is P. Each payment in the second year is $2P$. Each payment in the third year is $3P$. The payments continue to increase in the same pattern.

Determine P.

Solution:

$$A = B \text{ ==> } \frac{20,000}{\delta} = \frac{1000}{\delta^2} \text{ ==> } 20,000\delta = 1000 \text{ ==> } \delta = 0.05$$

$$PVofA = \frac{20,000}{\delta} = \frac{20,000}{0.05} = 400,000$$

$$PVofC = P\left(\frac{(1+i)}{i \cdot \frac{i^{(4)}}{4}}\right)$$

$$e^{\delta} = 1 + i \text{ ==> } i = 0.051271096$$

$$1 + i = \left(1 + \frac{i^{(4)}}{4}\right)^4 \text{ ==> } \frac{i^{(4)}}{4} = (1.051271096)^{1/4} - 1 = 0.01257845$$

$$400,000 = P\left(\frac{1.051271096}{(0.051271096)(0.01257845)}\right) \text{ ==> } P = 245.38$$

Flo is making payments continuously at a rate of $1000t + 500$ at time t for the next 10 years.

Using a discount function of $1 - 0.006t^2$, calculate the present value of Flo's payments.

Solution:

$$PV = \int_0^{10} (1000t + 500)(1 - 0.006t^2)dt$$

$$= \int_0^{10} (1000t + 500 - 6t^3 - 3t^2)dt$$

$$\left[500t^2 + 500t - \frac{6}{4}t^4 - t^3 \right]_0^{10} = 39,000$$

Kimberly purchases an 18 year continuous annuity that pays at a rate of $1300t$ at time t.

Calculate the present value of this annuity using a force of interest of $\delta = 0.0625$.

Solution:

$$1300(Ia)_{\overline{18|}} = 1300\left[\frac{\dfrac{1-e^{-18(0.0625)}}{0.0625} - 18e^{-18(0.0625)}}{0.0625}\right] = 103{,}205.78$$

Jordan is the beneficiary of a 12 year continuous annuity. The annuity pays at a rate of $300+10t$ at time t.

Using a discount function of $1-0.04t$, calculate the present value of Jordan's annuity.

Solution:

$$PV=\int_0^{12} f(t)v(t)dt=\int_0^{12}(300+10t)(1-0.04t)dt=\int_0^{12}300-12t+10t-0.4t^2\cdot dt=$$

$$\left[300t-t^2-\frac{0.4t^3}{3}\right]_0^{12}=3600-144-230.40=3225.60$$

Caroline has the option of the following two continuous annuities:

a. An annuity that pays continuously at an annual rate of 1000 for 20 years.

b. An annuity that pays continuously at a rate of Xt at time t for 20 years.

Both annuities have the same present value at a force of interest of 8%.

Determine X .

Solution:

$$Option\ a = Option\ b$$

$$1000\bar{a}_{\overline{20}|} = X\left[(\bar{I}\bar{a})_{\overline{20}|}\right]$$

$$1000\left(\frac{1-e^{-0.08(20)}}{0.08}\right) = X\left[\frac{\frac{1-e^{-0.08(20)}}{0.08} - 20e^{-0.08(20)}}{0.08}\right]$$

$$9976.29 = X(74.2295) ==> X = 134.40$$

A 20 year continuous annuity pays at the rate of $300+10t$ at time t.

Calculate the present value of this annuity at a force of interest of 7%.

Solution:

$$PV = 300\bar{a}_{\overline{20|}} + 10(\bar{I}\bar{a})_{\overline{20|}} =$$

$$300\left(\frac{1-e^{-(20)(0.07)}}{0.07}\right) + 10\left[\frac{\left(\frac{1-e^{-(20)(0.07)}}{0.07}\right) - 20e^{-(20)(0.07)}}{0.07}\right]$$

$$= 3228.87 + 832.99 = 4061.86$$

A continuous annuity pays at a rate of $1000t$ at time t for 30 years.

Calculate the present value of this annuity using $\delta = 0.10$.

Solution:

$$1000\left(\bar{I}\bar{a}\right)_{\overline{30}|} = 1000\left(\frac{\bar{a}_{\overline{30}|} - 30v^{30}}{\delta}\right) = 1000\left(\frac{\frac{1-e^{-30(0.1)}}{0.1} - 30e^{-30(0.1)}}{0.10}\right) = 80,085.17$$

A continuous perpetuity that pays at a rate of $1000t$ at time t has a present value of 25,000 when calculated at a force of interest of δ .

Chengjia is receiving a continuous 20 year annuity that pays at a rate of $500t$ at time t .

Calculate the present value of Chengjia's annuity using a force of interest equal to 0.5δ which is one half the force of interest used to calculate the present value of the perpetuity.

Solution:

We will use the perpetuity to find δ.

$$25,000 = \frac{1000}{\delta^2} \implies \delta^2 = \frac{1}{25} \implies \delta = 0.2$$

Now we will find the present value of the annuity at $0.5\delta = 0.10$

$$PV = (500)\left(\frac{\bar{a}_{\overline{20}|} - 20e^{-20(\delta)}}{\delta}\right) = (500)\left(\frac{\frac{1-e^{-20(0.1)}}{0.1} - 20e^{-20(0.1)}}{0.1}\right) = 29,699.71$$

Caroline has just been named the beneficiary of the Chen Trust Fund. The balance in the Chen Trust Fund is 1,000,000.

Caroline can choose one of the following options as her payout from the trust fund:

a. A continuous perpetuity that will pay at a rate of $10,000t$ at time t; or
b. An 30 year increasing annuity with quarterly payments at the end of each quarter. The first payment will be P. The second payment will be $2P$. The third payment will be $3P$. The payments will continue to increase in the same pattern until the last payment is made.

The two options have a present value of 1,000,000 based on a force of interest of δ.

Determine P.

Solution:

Option a:

$$1,000,000 = \frac{10,000}{\delta^2} ==> \delta = 0.1$$

Option b:

Since payments are quarterly, we need $\frac{i^{(4)}}{4}$. $e^{0.1} = \left(1+\frac{i^{(4)}}{4}\right)^4$.

$$\frac{i^{(4)}}{4} = e^{0.25} - 1 = 0.025315121$$

$$1,000,000 = Pa_{\overline{120}|} + \frac{P}{0.025315121}\left(a_{\overline{120}|} - 120(1.025315121)^{-120}\right)$$

$$P = \frac{1,000,000}{a_{\overline{120}|} + \frac{1}{0.025315121}\left(a_{\overline{120}|} - 120(1.025315121)^{-120}\right)} = 778.66$$