Topic: Duration

Shyam is receiving an annuity due with 12 annual payments of 200.

Calculate the Macaulay Duration of Shyam's annuity using an annual effective interest rate of 8%.

Solution:

$$MacDur = \frac{\sum C_t(t)v^t}{\sum C_t v^t} = \frac{(200)(0)v^0 + (200)(1)v^1 + ... + (200)(11)v^{11}}{200\ddot{a}_{\overline{12|}}}$$

$$\frac{(200)v^1 + ... + (2200)(11)v^{11}}{200\ddot{a}_{\overline{12|}}} = \frac{200a_{\overline{11|}} + \frac{200}{0.08}(a_{\overline{11|}} - 11(1.08)^{-11})}{200\ddot{a}_{\overline{12|}}} =$$

$$\frac{200\left(\frac{1-(1.08)^{-11}}{0.08}\right) + \frac{200}{0.08}\left(\frac{1-(1.08)^{-11}}{0.08} - 11(1.08)^{-11}\right)}{200\left(\frac{1-(1.08)^{-12}}{0.08}\right)(1.08)} = 4.5957$$

Alex owns a 15 year bond with annual coupons of 100 and a maturity value of 2000.

Calculate the Modified Duration of her bond using an annual effective interest rate of 5%.

Solution:

$$ModDur = v\frac{\sum C_t(t)v^t}{\sum C_t v^t} = v\left(\frac{(100)(1)v^1 + (100)(2)v^2 + \ldots + (100)(15)v^{15} + (2000)(15)v^{15}}{100a_{\overline{15|}} + 2000v^{15}}\right)$$

$$v\left(\frac{100a_{\overline{15|}} + \frac{100}{0.05}(a_{\overline{15|}} - 15(1.05)^{-15}) + (2000)(15)(1.05)^{-15}}{100a_{\overline{15|}} + 2000v^{15}}\right) =$$

$$(1.05)^{-1}\left[\frac{100\left(\frac{1-(1.05)^{-15}}{0.05}\right) + \frac{100}{0.05}\left(\frac{1-(1.05)^{-15}}{0.05} - 15(1.05)^{-15}\right)}{100\left(\frac{1-(1.05)^{-15}}{0.05}\right) + 2000(1.05)^{-15}}\right] = 10.38$$

Matt owns a 20 year bond with annual coupons of 1000 and a maturity value of 20,000.

Calculate the Modified Duration of this bond at an annual effective interest rate of 8%.

Solution:

$$MacDur = \frac{\Sigma C_t(t)v^t}{\Sigma C_t v^t}$$

$$= \frac{1{,}000(1)v^1 + 1{,}000(2)v^2 + 1{,}000(3)v^3 + \cdots + 1{,}000(20)v^{20} + 20{,}000(20)v^{20}}{1{,}000\, a_{\overline{20}|0.08} + 20{,}000(1.08)^{-20}}$$

$$= \frac{1{,}000v^1 + 2{,}000v^2 + 3{,}000v^3 + \cdots + 20{,}000v^{20} + 400{,}000v^{20}}{1{,}000\, a_{\overline{20}|0.08} + 20{,}000(1.08)^{-20}}$$

$$= \frac{1{,}000(1.08)^{-1} + 2{,}000(1.08)^{-2} + 3{,}000(1.08)^{-3} + \cdots + 20{,}000(1.08)^{-20} + 400{,}000(1.08)^{-20}}{1{,}000\, a_{\overline{20}|0.08} + 20{,}000(1.08)^{-20}}$$

$$= \frac{1{,}000\left(\frac{1-(1.08)^{-20}}{0.08}\right) + \left(\frac{1{,}000}{0.08}\right)\left(\frac{1-(1.08)^{-20}}{0.08} - 20(1.08)^{-20}\right) + 400{,}000(1.08)^{-20}}{1{,}000\left(\frac{1-(1.08)^{-20}}{0.08}\right) + 20{,}000(1.08)^{-20}}$$

$$= 11.67523699$$

$$ModDur = v(MacDur) = (1.08)^{-1}(11.6752) = 10.8104$$

A 20 year bond matures for its par value of 10,000. The bond pays annual coupons at a rate of 7%.

Calculate the Modified duration of the bond using an interest rate of 5%.

Solution:

$$Coupon = 10,000(0.07) = 700$$

N=20, PMT=700, FV=10,000, I/Y=5, CPT PV=12,492.44207 = Price of Bond

$$MacDur = \frac{\sum C_t \cdot t \cdot v^t}{\sum C_t \cdot v^t} = \frac{(700)(1)v^1 + (700)(2)v^2 + \ldots + (700)(20)v^{20} + (10,000)(20)v^{20}}{\text{Price of Bond}}$$

$$= \frac{700a_{\overline{20|}} + \frac{700}{0.05}(a_{\overline{20|}} - 20v^{20}) + (10,000)(20)v^{20}}{12,492.44207} = \frac{153,043.3334}{12,492.44207} = 12.25087397$$

$$ModDur = v(MacDur) = \frac{12.25087397}{1.05} = 11.6675$$

An annuity due pays 1000 at the beginning of each year for 22 years.

Determine the Macaulay Duration of this annuity at an annual effective interest rate of 6%.

Solution:

$$MacDur=\frac{\sum C_t(t)v^t}{\sum C_t v^t}=\frac{(1000)(0)v^0+(1000)(1)v^1+\ldots+(1000)(21)v^{21}}{1000\ddot{a}_{\overline{22}|}}$$

$$\frac{(1000)v^1+\ldots+(21,000)(21)v^{21}}{1000\ddot{a}_{\overline{22}|}}=\frac{1000a_{\overline{21}|}+\frac{1000}{0.06}(a_{\overline{21}|}-21(1.06)^{-21})}{1000\ddot{a}_{\overline{22}|}}=$$

$$\frac{1000\left(\frac{1-(1.06)^{-21}}{0.06}\right)+\frac{1000}{0.06}\left(\frac{1-(1.06)^{-21}}{0.06}-21(1.06)^{-21}\right)}{1000\left(\frac{1-(1.06)^{-22}}{0.06}\right)(1.06)}=8.217$$

Jake will pay Yash 100 at the end of one year, 200 at the end of two years, 300 at the end of three years, and 400 at the end of four years.

Calculate the Modified Duration of these payments at an interest rate of 10%.

Solution:

$$ModDur = v\frac{\sum C_t (t) v^t}{\sum C_t v^t}$$

$$= (1.10)^{-1}\left(\frac{(100)(1)(1.10)^{-1} + (200)(2)(1.10)^{-2} + (300)(3)(1.10)^{-3} + (400)(4)(1.10)^{-4}}{(100)(1.10)^{-1} + (200)(1.10)^{-2} + (300)(1.10)^{-3} + (400)(1.10)^{-4}}\right)$$

$$= 2.638$$

An annuity due pays 20,000 at the beginning of each year for 20 years.

Calculate the Modified Duration of this annuity at an annual effective interest rate of 8%.

Solution:

$$ModDur = v\frac{\sum C_t (t) v^t}{\sum C_t v^t}$$

$$= (1.08)^{-1}\frac{(20,000)(0)(1.08)^0 + (20,000)(1)(1.08)^1 + \ldots + (20,000)(19)(1.08)^{19}}{(20,000)(1.08)^0 + (20,000)(1.08)^1 + \ldots + (20,000)(1.08)^{19}}$$

$$= (1.08)^{-1}\frac{(20,000)(1)(1.08)^1 + \ldots + (20,000)(19)(1.08)^{19}}{(20,000)(1.08)^0 + (20,000)(1.08)^1 + \ldots + (20,000)(1.08)^{19}}$$

$$= (1.08)^{-1}\frac{(20,000)\left(\frac{1-(1.08)^{-19}}{0.08}\right) + \left(\frac{20,000}{0.08}\right)\left(\frac{1-(1.08)^{-19}}{0.08} - 19(1.08)^{-19}\right)}{(20,000)\left(\frac{1-(1.08)^{-20}}{0.08}\right)(1.08)}$$

$$= (1.08)^{-1}\frac{1,492,339.48}{212,071.984} = 6.5157$$

A 25 year bond pays semi-annual coupons of 200 and matures for 1000.

Calculate the modified duration of this bond at an annual effective interest rate of 12.36%.

Solution:

$$MacDur = \frac{\sum C_t (t) v^t}{\sum C_t v^t}$$

$$= \frac{200(0.5)v^{0.5} + 200(1)v + 200(1.5)v^{1.5} + ... + (200)(25)v^{25} + (1000)(25)v^{25}}{200a_{\overline{50}|0.06} + 1000(1.06)^{-50}}$$

$$= \frac{100v^{0.5} + 200v^{1} + 300v^{1.5} + ... + 5000v^{25} + (25,000)v^{25}}{200a_{\overline{50}|0.06} + 1000(1.06)^{-50}}$$

$$= \frac{100(1.06)^{-1} + 200(1.06)^{-2} + 300(1.06)^{-3} + ... + 5000(1.06)^{-50} + (25,000)(1.1236)^{-25}}{200a_{\overline{50}|0.06} + 1000(1.06)^{-50}}$$

$$= \frac{100\left(\frac{1-(1.06)^{-50}}{0.06}\right) + \left(\frac{100}{0.06}\right)\left(\frac{1-(1.06)^{-50}}{0.06} - 50(1.06)^{-50}\right) + (25,000)(1.1236)^{-25}}{200a_{\overline{50}|0.06} + 1000(1.06)^{-50}}$$

$$= 7.69621$$

$$ModDur = v(MacDur) = (1.1236)^{-1}(7.69621) = 6.8496$$

A 30 year bond has annual coupons of 1000 and a maturity value of 20,000.

Calculate the Modified Duration of this bond at an annual effective interest rate of 6%.

Solution:

$$ModDur = v(MacDur) = v\frac{\sum C_t(t)v^t}{\sum C_t v^t}$$

$$= v\,\frac{(1000)(1)v^1 + (1000)(2)v^2 + \ldots + (1000)(30)v^{30} + (20,000)(30)v^{30}}{1000a_{\overline{30|}} + 20,000(1.06)^{-30}}$$

$$= \frac{1000a_{\overline{30|}} + \frac{1000}{0.06}\left(a_{\overline{30|}} - 30(1.06)^{-30}\right) + (20,000)(30)v^{30}}{(17,247.03)(1.06)} = 14.25$$

A 30 year bond matures for 10,000 and has semi-annual coupons of 300.

Calculate the Modified duration at an annual effective interest rate of 10.25%.

Solution:

$$\frac{i^{(2)}}{2} = (1.1025)^{0.5} - 1 = 0.05$$

$$\text{Price} = 300\left(\frac{1-(1.05)^{-60}}{0.05}\right) + 10,000(1.05)^{-60} = 6214.14$$

Modified Duration =

$$\frac{\sum C_t t v^t}{\sum C_t v^t} v = \frac{300(0.5)v^{0.5} + 300(1)v + 300(1.5)v^{1.5} + \ldots + 300(30)v^{8} + (10,000)(30)v^{30}}{6214.14}\left(\frac{1}{1.1025}\right)$$

$$= \frac{150(1.1025)^{-0.5} + 300(1.1025)^{-1} + 450(1.1025)^{-1.5} + \ldots + 9000(1.1025)^{-30} + (10,000)(30)v^{30}}{6214.14}\left(\frac{1}{1.1025}\right)$$

$$= \frac{150(1.05)^{-1} + 300(1.05)^{-2} + 450(1.05)^{-3} + \ldots + 9000(1.05)^{-60} + 10,000(30)v^{30}}{6214.14}\left(\frac{1}{1.1025}\right)$$

$$= \frac{150a_{\overline{60}|0.05} + \frac{150}{0.05}\left(a_{\overline{60}|0.05} - 60\left(1.05\right)^{-60}\right) + (300,000)\left(1.1025\right)^{-30}}{(6214.14)\left(1.1025\right)}$$

$$= \frac{150\left(\frac{1-(1.05)^{-60}}{0.05}\right) + \frac{150}{0.05}\left(\left(\frac{1-(1.05)^{-60}}{0.05}\right) - 60(1.05)^{-60}\right) + 300,000\left(1.1025\right)^{-30}}{(6214.14)\left(1.1025\right)} = 9.641$$

A perpetuity due pays 100 at the start of each year.

Calculate the Macaulay Duration at an interest rate of 6%.

Solution:

$$MacDur = \frac{\sum C_t (t) v^t}{\sum C_t v^t} = \frac{(100)(0)v^0 + (100)(1)v^1 + (100)(2)v^2 + \ldots}{(100)v^0 + (100)v^1 + (100)v^2 + \ldots}$$

$$= \frac{(100)(1)v^1 + (100)(2)v^2 + \ldots}{(100)v^0 + (100)v^1 + (100)v^2 + \ldots} = \frac{\frac{100}{i} + \frac{100}{i^2}}{\frac{100}{i}(1+i)} = \frac{1 + \frac{1}{i}}{(1+i)}$$

$$= \frac{1 + \frac{1}{0.06}}{1.06} = 16.67$$