Topic: Force of Interest

Brett has the option to invest in either Account A or Account B:

a. Account A pays a nominal annual interest rate of 6% compounded monthly for ten years.

b. Account B pays a rate equivalent to $d^{(2)}$ for the first six years and then earns a force of interest of $\delta_t = 0.008t$ for the next four years.

If Brett invests 1000 in either account, he will have the same amount at the end of 10 years.

Determine $d^{(2)}$.

Solution:

$$\text{Account A} = 1000\left(1+\frac{0.06}{12}\right)^{(10)(12)} = 1819.40$$

$$\text{Account B} = 1000\left(1-\frac{d^{(2)}}{2}\right)^{-(6)(2)}\left(e^{\int_6^{10} 0.008t \cdot dt}\right) = 1819.40$$

$$\Longrightarrow \left(1-\frac{d^{(2)}}{2}\right)^{-12}\left(e^{0.004t^2\big]_6^{10}}\right) = 1.81940 \Longrightarrow \left(1-\frac{d^{(2)}}{2}\right)^{-12}\left(e^{0.4-0.144}\right) = 1.81940$$

$$\Longrightarrow \left(1-\frac{d^{(2)}}{2}\right)^{-12} = \frac{1.81940}{e^{0.256}} = 1.40847 \Longrightarrow d^{(2)} = 2\left[1-(1.40847)^{-1/12}\right] = 0.056277$$

Dylan invests 13,000 today in an account at Nick Bank. Dylan also invests another 5000 in the same account at the end of 5 years.

The account at Nick Bank earns a force of interest of $0.08+0.002t$ where t is measured from today.

Determine how much Dylan has at the end of 9 years.

Solution:

$$Amount = 13,000e^{\int_0^9 (0.08+0.002t)dt} + 5000e^{\int_5^9 (0.08+0.002t)dt}$$

$$= 13,000e^{\left[0.08t+0.001t^2\right]_0^9} + 5000e^{\left[0.08t+0.001t^2\right]_5^9} =$$

$$13,000e^{\left[0.08(9)+0.001(9)^2-0\right]} + 5000e^{\left[0.08(9)+0.001(9)^2-0.08(5)-0.001(5)^2\right]}$$

$$13,000e^{0.801} + 5000e^{0.376} = 36,243.21$$

Alisa invests 42,000 in Amir Bank. At the end of 10 years, Alisa has 100,000.

Amir Bank pays interest based on the following:

a. The first two years, Amir pays an annual effective interest rate of i .

b. During the next three years, Amir pays a nominal discount rate of 8% compounded quarterly.

c. During the last five years, Amir pays a force of interest equal to $\delta_t = 0.04 + 0.001t^2$ where t is measured from the date of the original investment of 42,000.

Determine i .

Solution:

$$(42,000)(1+i)^2\left(1-\frac{0.08}{4}\right)^{-4(3)} e^{\int_5^{10}\left(0.04+0.001r^2\right)dr} = 100,000$$

$$\boxed{e^{\int_5^{10}\left(0.04+0.001r^2\right)dr} = e^{\left[0.04r+\frac{0.001}{3}r^3\right]_5^{10}} = e^{(0.4+0.333333333-0.2-0.0411666666)} = e^{0.49166666}}$$

$$(1+i)^2 = \frac{100,000}{(42,000)\left(1-\frac{0.08}{4}\right)^{-4(3)} e^{0.49166666}} = 1.14708605$$

$$i = (1.14708605)^{0.5} - 1 = 0.068975493$$

You are given that $v(t) = \left[1+\beta t^2\right]^{-1}$.

You are also given that $\delta_5 = \delta_{10}$.

Determine β .

Solution:

$$v(t) = \frac{1}{1+\beta t^2} = \frac{1}{a(t)} ==> a(t) = 1+\beta t^2$$

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{2\beta t}{1+\beta t^2}$$

$$\delta_5 = \delta_{10} ==> \frac{2\beta(5)}{1+\beta(5)^2} = \frac{2\beta(10)}{1+\beta(10)^2} ==> 10\beta\left[1+100\beta\right] = 20\beta\left[1+25\beta\right]$$

$$==> \left[1+100\beta\right] = 2\left[1+25\beta\right] ==> 1+100\beta = 2+50\beta$$

$$50\beta = 1 ==> \beta = 0.02$$

You are given that $a(t)=\alpha+\beta t^2$ and that $\delta_{10}=0.10$.

Shina invests 1000 at time zero using the above accumulation function.

How much does Shina have after 20 years?

Solution:

$a(0)=1 ==> \alpha=1$

$$\delta_t=\frac{a'(t)}{a(t)}=\frac{2\beta t}{1+\beta t^2} ==> \delta_{10}=0.10=\frac{2\beta(10)}{1+\beta(10)^2} ==> (0.10)(1+100\beta)=20\beta$$

$$==> 0.1+10\beta=20\beta ==> 0.1=10\beta ==> \beta=0.01$$

$$\text{Amount after 20 years}=1000a(20)=1000[1+0.01(20)^2]=5000$$

Alex invests 10,000 in an account earning simple interest at a rate of 8%.

Quinn invests S in an account earning 6% compounded continuously.

At the end of 10 years, Alex and Quinn have the same amount.

Yash invests S at an annual interest rate equivalent to a discount rate of 7%.

Determine the amount that Yash will have at the end of 10 years.

Solution:

$$Alex: 10,000[1+0.08(10)] = 18,000$$

$$Quinn: Se^{(0.06)(10)} = 18,000 ==> S = \frac{18,000}{e^{0.6}} = 9878.60945$$

$$Yash: 9878.60945(1-0.07)^{-10} = 20,411.10$$

You are given:

a. $v(t)=\dfrac{1}{\beta+0.01t+\alpha t^2}$

b. $i_{11}=0.04$

Calculate δ_{10} .

Solution:

$$v(t)=\frac{1}{a(t)} \Longrightarrow a(t)=\beta+0.01t+\alpha t^2$$

$$a(0)=1 \Longrightarrow \beta+0.01(0)+\alpha(0^2)=1 \Longrightarrow \beta=1$$

$$i_{11}=0.04 \Longrightarrow \frac{a(11)-a(10)}{a(10)}=0.04 \Longrightarrow \frac{1+0.01(11)+\alpha(11^2)-[1+0.01(10)+\alpha(10^2)]}{1+0.01(10)+\alpha(10^2)}=0.04$$

$$\Longrightarrow \frac{0.11+121\alpha-0.10-100\alpha}{1+0.1+100\alpha}=0.04 \Longrightarrow 0.01+21\alpha=0.04+0.004+4\alpha$$

$$\Longrightarrow 17\alpha=0.034 \Longrightarrow \alpha=0.002$$

$$a(t)=1+0.01t+0.002t^2$$

$$\delta_t=\frac{a'(t)}{a(t)}=\frac{0.01+0.004t}{1+0.01t+0.002t^2} \Longrightarrow \delta_{10}=\frac{0.01+0.004(10)}{1+0.01(10)+0.002(10^2)}=0.03846$$

You are given that $a(t) = \alpha + \beta t^3$. You are also given that $\delta_{10} = 0.25$.

Calculate i_5.

Solution:

$a(0) = 1 ==> \alpha + \beta(0) = 1 ==> \alpha = 1$

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{3\beta t^2}{1+\beta t^3} ==> \delta_{10} = 0.25 = \frac{3\beta(10)^2}{1+\beta(10)^3} ==> 0.25 + 250\beta = 300\beta$$

$$==> 0.25 = 50\beta ==> \beta = 0.005$$

$$i_5 = \frac{a(5)-a(4)}{a(4)} = \frac{1+(0.005)(5)^3 - [1+0.005(4)^3]}{1+0.005(4)^3} = 0.23106$$

Tom invests 10,000 in the Pham Fund for 10 years.

a. The Pham Fund pays simple interest at an annual rate of 5% during the first 3 years.

b. The Pham Fund pays an interest rate equivalent to a discount rate of 8% convertible quarterly during the next five years.

c. The Pham Fund pays a force of interest of 10% during the last two years of Tom's investment.

Determine the amount of money that Tom has at the end of 10 years.

Solution:

$$Amount = (10,000)[1+(0.05)(3)]\left[1-\frac{0.08}{4}\right]^{-(4)(5)}\left[e^{(0.1)(2)}\right] = 21,039.49$$

You are given that $v(t) = \dfrac{1}{1+0.002t^2}$.

Calculate $1000(\delta_5 - d_5)$.

Solution:

$$v(t) = \frac{1}{1+0.002t^2} \text{ ==> } a(t) = 1+0.002t^2$$

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{0.004t}{1+0.002t^2} \text{ ==> } \delta_5 = \frac{(0.004)(5)}{1+0.002(5)^2} = \frac{0.02}{1.05} = 0.01905$$

$$d_5 = \frac{a(5)-a(4)}{a(5)} = \frac{1+0.002(5)^2 - \left[1+0.002(4)^2\right]}{1+0.002(5)^2} = \frac{1.05-1.032}{1.05} = 0.01714$$

$$Answer = (1000)(0.01905 - 0.01714) = 1.91$$

You are given:

a. $v(t) = \dfrac{1}{\alpha + \beta t^2}$

b. $\delta_5 = 0.08$

Peyton invests X today in an account and has 25,000 at the end of 10 years.

Determine X.

Solution:

$$v(t) = \frac{1}{a(t)} ==> a(t) = \alpha + \beta t^2$$

$$a(0) = 1 ==> \alpha = 1$$

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{2\beta t}{1+\beta t^2} ==> \delta_5 = \frac{10\beta}{1+25\beta} = 0.08$$

$$10\beta = 0.08 + 2\beta ==> 8\beta = 0.08 ==> \beta = \frac{0.08}{8} = 0.01$$

$$Xa(10) = 25,000 ==> X = \frac{25,000}{a(10)} = \frac{25,000}{1+0.01(10)^2} = \frac{25,000}{2} = 12,500$$

Adam has the choice of the following car loans:

a. Bradley Bank will loan Adam 13,000 for five years to be repaid at an annual interest rate of 8% compounded continuously.

b. Chen Bank will loan Adam 13,000 for five years to be repaid at an annual interest rate equivalent to a discount rate of 8% compounded monthly.

State which loan Adam should choose. Demonstrate that the loan that you chose is better for Adam than the other loan.

Solution:

Under Option A:

$$\text{Amount to be Repaid} = (13,000)(e^{(0.08)(5)}) = 19,393.72$$

Under Option B:

$$\text{Amount to be Repaid} = (13,000)\left(1 - \frac{0.08}{12}\right)^{-12(5)} = 19,419.71$$

Adam should choose the loan that will result in the smallest repayment so he should choose Option A which is the loan from Bradley Bank.

Claire invests 20,000 today in an account earning a force of interest of $\delta_t = 0.02 + 0.01t$.

How much does Claire have at the end of 10 years.

Solution:

$$Amount = 20,000e^{\int_0^{10}(0.02+0.01t)dt}$$

$$= 20,000e^{0.02t+0.005t^2\Big|_0^{10}}$$

$$= 20,000e^{0.7}$$

$$= 20,000(2.01375)$$

$$= 40,275.05$$