Topic: Non-Level Annuities, Arithmetic Series

Sharruna invests 20,000 in the Tang Fund at the end of each year for 13 years. The Tang Fund earns an annual effective interest rate of 7%.

At the end of each year, Sharruna takes the interest earned in the Tang Fund and moves it into the Vandokkenburg Fund. Vandokkenburg earns and annual effective interest rate of 9% per year.

Determine the amount that Sharruna has at the end of 13 years taking into account both the Tang Fund and the Vandokkenburg Fund.

Solution:

In the first year, there is no interest transferred to the Vandokkenburg Fund as there is no money invested in the Tang Fund the first year. At the end of the second year, 1400 is withdrawn from Tang and invested in Vandokkenburg. At the end of the third year, 2800 is withdrawn from Tang and invested in Vandokkenburg. The amount continues to increase each year.

At the end of 13 years, there will be (13)(20,000) = 260,000 in the Tang Fund

At the end of 13 years, the value of the Vandokkenburg Fund will be:

$$\left[1400a_{\overline{12}|} + \frac{1400}{0.09}\left(a_{\overline{12}|} - 12(1.09)^{-12}\right)\right](1.09)^{12} = 154,830.43$$

$$Total = 260,000 + 154,830.43 = 414,830.43$$

Sid wants to provide an increasing annuity for his mother for the next 20 years. The annuity will make annual payments at the beginning of each year. The first payment today will be 1000. The second payment will be 3000. The third payment will be 5000. The payments will continue in the same pattern each payment being 2000 greater than the prior payment.

Using an annual effective interest rate of 6.25%, calculate the present value of the annuity.

Solution:

$$PV = \left[1000a_{\overline{20}|} + \left(\frac{2000}{0.0625}\right)\left(a_{\overline{20}|} - 20(1.0625)^{-20}\right)\right](1.0625)$$

$$= \left[1000\left(\frac{1-(1.0625)^{-20}}{0.0625}\right) + \left(\frac{2000}{0.0625}\right)\left(\frac{1-(1.0625)^{-20}}{0.0625} - 20(1.0625)^{-20}\right)\right](1.0625)$$

$$= 191{,}858.39$$

We multiply the P&Q formula above by (1.0625) since this is an annuity due.

Peggy is receiving an annuity immediate with monthly payments for the next 20 years. The first payment is 100. The second payment is 200. The third payment is 300. The payments continue increase until the last payment is 24,000.

Peggy takes each payment and invests it in the Tang Fund. The Tang Fund pays an interest rate of 9% compounded monthly.

Determine the amount that Peggy will have at the end of 20 years.

Solution:

Every payment is increasing so we have the P&Q formula.

$$i^{(12)} = 0.09 \Rightarrow \frac{i^{(12)}}{12} = \frac{0.09}{12} = 0.0075$$

$$AV = \left[100a_{\overline{240|}} + \frac{100}{0.0075}\left(a_{\overline{240|}} - 240v^{240}\right)\right](1.0075)^{240}$$

$$= \left[100\left(\frac{1-(1.0075)^{-240}}{0.0075}\right) + \frac{100}{0.0075}\left(\left(\frac{1-(1.0075)^{-240}}{0.0075}\right) - 240(1.0075)^{-240}\right)\right](1.0075)^{240}$$

$$= (960,526.11)(1.0075)^{240}$$

$$= 5,771,946.95$$

Nathan has won the lottery! The lottery will make annual payments at the beginning of each year for the next 25 years. The first payment is 100,000. The second payment is 150,000. The third payment is 200,000. The same pattern continues with each payment being 50,000 more than the prior payment until a payment of 1,300,000 is made at the start of the 25th year.

Calculate the present value of these payments at an annual effective interest rate of 9%.

Solution:

This is the P&Q Formula. We note that it is an annuity due.

$$PV = \left[100,000 a_{\overline{25|}} + \frac{50,000}{0.09}(a_{\overline{25|}} - 25(1.09)^{-25})\right](1.09)$$

$$a_{\overline{25|}} \frac{1-(1.09)^{-25}}{0.09} = 9.822579605$$

$$PV = 5,263,154.65$$

Kate invests 2500 into the Adams Fund at the end of each year for the next 13 years. The Adams Fund earns an annual effective interest rate of 7%.

At the end of each year, the interest earned in the Adams Fund is transferred to the Baker Fund. The Baker fund earns an annual effective interest rate of 8%.

Determine the amount that Kate will have at the end of 13 years when she combines the amount in the Adams Fund and the amount in the Baker Fund.

Solution:

There will be 13 payments of 2500 deposited into the Adams Fund. Since the interest from the Adams Fund is withdrawn each year, the amount in the Adams Fund at the end of 13 years is just the money deposited which will be (13)(2500) = 32,500.

Each year, the interest earned by the Adams Fund will be transfered to the Baker Fund. The amount transfered to Baker Fund is 175 at the of year 2, 350 at the end of year 3, etc. Note that there is no transfer at the end of year 1 because there was no money in Adams Fund during the first year. The Baker Fund earns 8% so the amount in the Baker Fund at end of 13 years is:

$$AV = \left[175a_{\overline{12|}} + \frac{175}{0.08}(a_{\overline{12|}} - 12(1.08)^{-12}\right](1.08)^{12} = 18,583.46$$

$$a_{\overline{12|}} = \frac{1-(1.08)^{-12}}{0.08} = 7.536078017$$

$$Total = 32,500 + 18,583.46 = 51,083.46$$

Assad invests money invests 10,000 at the beginning of the each year for 8 years into Fund A. Fund A pays an annual effective interest rate of 6%.

At the end of each year, Assad withdraws the interest earned in Fund A and deposits into Fund B which is earning an annual effective interest rate of 8.25%.

Determine the total amount that Assad will have at the end of 8 years taking into account both Fund A and Fund B.

Solution:

Fund A pays all interest to Fund B. Therefore, at the end of 8 years, all that is left in Fund A are the deposits which are $(10,000)(8)=80,000$. The interest paid into Fund B is $(10{,}000)(0.06)=600$ at the end of the first year, $(2)(10{,}000)(0.06)=1200$ at the end of the second year, ... until $(8)(10{,}000)(0.06)$ is paid at the end of the 8th year. The amount in Fund B at the end of 8 years using the P&Q formula is:

$$\left[600a_{\overline{8}|0.0825}+\frac{600}{0.0825}\left(a_{\overline{8}|0.0825}-8(1.0825)^{-8}\right)\right](1.0825)^{8}$$

$$\left[600\left[\frac{1-(1.0825)^{-8}}{0.0825}\right]+\frac{600}{0.0825}\left(\left[\frac{1-(1.0825)^{-8}}{0.0825}\right]-8(1.0825)^{-8}\right)\right](1.0825)^{8}=26,317.42$$

$$Total=80,000+26,317.42=106,317.42$$

Eston will buy one of the following set of payments:

a. A perpetuity with increasing payments. The payments are 100 at the end of the first year, 1100 at the end of the second year, 2100 at the end of the third year, etc. Payments continue forever with each payment being 1000 greater than the previous payment.

b. A 20 year annuity due with annual increasing payments. The first payment is Q. The second payment is $Q(1.08)$. The third payment is $Q(1.08)^2$. The payments continue for 20 years with each payment being 108% of the prior payment.

Both the perpetuity and annuity have a present value of 249,575.30 at an interest rate of i.

Determine Q.

Solution:

$$PV = \frac{100}{i} + \frac{1000}{i^2} = 249,575.30$$

$$100i + 1000 = 249,575.30i^2 \implies 249,575.30i^2 - 100i - 1000 = 0$$

$$i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-100) \pm \sqrt{(-100)^2 - 4(249,575.30)(-1000)}}{(2)(249,575.30)} = 0.0635$$

$$PVii = Q + Q(1.08)(1.0635)^{-1} + ... + Q(1.08)^{19}(1.0635)^{-19}$$

$$= \frac{Q - Q(1.08)^{20}(1.0635)^{-20}}{1-(1.0635)(1.08)^{-1}} = 10,738.46$$

Leen is repaying a loan with quarterly payments for 5 years. The first payment is 1000. The second payment is 1600. The third payment is 2200. The payments continue to increase in the same pattern until the last payment is made.

The loan has an interest rate of 8% compounded quarterly.

Determine the amount of the loan. (The amount of the loan is the present value of the payments.)

Solution:

Every payment is increasing so we have the p&Q formula.

$$i^{(4)} = 0.08 \Longrightarrow \frac{i^{(4)}}{4} = 0.02$$

$$PV = 1000a_{\overline{20}|} + \frac{600}{0.02}\left[a_{\overline{20}|} - 20v^{20}\right]$$

$$= (1000)\left(\frac{1-(1.02)^{-20}}{0.02}\right) + \frac{600}{0.02}\left[\left(\frac{1-(1.02)^{-20}}{0.02}\right) - 20(1.02)^{-20}\right]$$

$$= 103{,}111.63$$

Ram invests 5000 at the beginning of each year into Fund A for 25 years. Fund A earns an annual effective interest rate of 10%. Each year, Ram removes the interest earned from Fund A and deposits it into Fund B which earns an annual interest rate of 8%.

Calculate the total amount (in both Funds) that Ram will have after 25 years.

Solution:

$$\text{Fund A } = (25)(5000) = 125,000$$

Fund A earns 500 of interest the first year, 1000 of interest the second year, etc.

$$\text{Fund B} = \left[500 a_{\overline{25}|0.08} + \frac{500}{0.08}\left(a_{\overline{25}|0.08} - 25(1.08)^{-25} \right) \right](1.08)^{25}$$

$$= \left[500\left(\frac{1-(1.08)^{-25}}{0.08} \right) + \frac{500}{0.08}\left(\left(\frac{1-(1.08)^{-25}}{0.08} \right) - 25(1.08)^{-25} \right) \right](1.08)^{25} = 337,215.09$$

$$\text{Total} = \text{Fund A} + \text{Fund B} = 125,000.00 + 337,215.04 = 462,215.09$$

Haozhe has 420,000 that he wants to invest. He has the following possible investments:

a. Purchase a perpetuity immediate with annual payments for 420,000. The perpetuity will pay 1000 at the end of the first year, 2000 at the end of the second year, 3000 at the end of the third year,etc.

b. Make a loan of 420,000 to Jun. Jun will repay the loan with level annual payments of 30,000 followed by a drop payment.

The annual effective interest rate for both investments is the same.

Calculate the amount of the drop payment.

Solution:

First determine the interest rate to be sued using Part a.

$$\frac{1000}{i}+\frac{1000}{i^2}=420,000 ==> 1000i+1000=420,000i^2 ==> i+1=420i^2$$

$$420i^2-i-1=0 ==> i=\frac{-(-1)\pm\sqrt{(-1)^2-(4)(420)(-1)}}{2(420)}=0.05$$

Now we find the drop payment:

$$\boxed{PV}\leftarrow 420,000;\boxed{I/Y}\leftarrow 5;\boxed{PMT}\leftarrow -30,000;\boxed{CPT}\boxed{N}\Rightarrow 24.676$$

$$\boxed{2nd}\boxed{Amort}\boxed{P1}\leftarrow 1;\boxed{P2}\leftarrow 24;\boxed{\downarrow}\boxed{Bal}\Rightarrow 19,482.01$$

$$Drop=(19,482.01)(1.05)=20,456.11$$

The Huang Company invests 10,000 at the end of each year with DeWitt Bank. DeWitt Bank pays an annual effective interest rate of 6%.

At the end of each year, Huang withdraws the interest earned from DeWitt and reinvests it in the Carvajal Fund which pays an annual effective interest rate of 8.5%.

Determine the total amount that Huang has at the end of 15 years.

Solution:

The first year, there is no interest transfered to Carvajal as there is no money invested in DeWitt the first year. At the end of the second year, 600 is withdrawn from DeWitt and invested in Carvajal. At the end of the third year, 1200 is withdrawn from DeWittt and invested in Carvajal. This amount contines to increase each year.

At the end of 15 years, there will be $(15)(10,000) = 150,000$ in the DeWitt Bank.

At the end of 15 years, there will be

$$\left[600a_{\overline{14|}} + \frac{600}{i}\left(a_{\overline{14|}} - 14(1.085)^{-14}\right)\right](1.085)^{14} = 93,403.97$$

$$Total = 150,000 + 93,403.97 = 243,403.97$$

Sam invests money quarterly into the Zhang Investment Fund. Zhang pays an annual effective interest rate of 6%.

Sam's invests 100 at the end of each quarter in the first year. He invests 200 at the end of each quarter in the second year. He invests 300 at the end of each quarter in the third year. This pattern continues until he invests 2000 at the end of each quarter in the 20th year.

Determine the amount that Sam will have in the Zhang Investment Fund at the end of the 20th year.

Solution:

$$AV = 100\left[\frac{\ddot{a}_{\overline{20|}0.06} - 20(1.06)^{-20}}{\frac{i^{(4)}}{4}}\right](1.06)^{20}$$

$$1.06 = \left(1+\frac{i^{(4)}}{4}\right)^4 \implies \frac{i^{(4)}}{4} = (1.06)^{1/4} - 1 = 0.014673846$$

$$AV = 100\left[\frac{\left(\frac{1-(1.06)^{-20}}{0.06}\right)(1.06) - 20(1.06)^{-20}}{0.014673846}\right](1.06)^{20} = 129,432.51$$

An annuity makes payments at the end of each year. The first payment is 10,000. The second payment is 9980. The third payment is 9960. The payments continue until the last payment is 8000.

Using an annual effective interest rate of 8%, calculate the present value of this annuity.

Solution:

The hard part here is determining the number of payments. There are 101 payments. One way to see this that there are there are 100 decreases (10000-8000)/20=100. But this does not include the first payment so there are 101 payments.

$$PV = 10,000a_{\overline{101|}} - \frac{20}{0.08}\left(a_{\overline{101|}} - 101(1.08)^{-101}\right)$$

$$a_{\overline{101|}} = \frac{1-(1.08)^{-101}}{0.08} = 12.49473849$$

$$PV = 121,834.33$$

Megan invests 5000 in the Stanley Fund at the end of each year for 20 years. The Stanley Fund pays an annual effective interest rate of 6%.

At the end of each year, any interest in the Stanley Fund is withdrawn and invested into the Hahn Fund. The Hahn Fund pays an annual effective interest rate of 5%.

Determine the amount that Megan has at the end of 20 years in both funds combined.

Solution:

Megan will deposit 5000 at the end of each year for 20 years. Since the interest will be removed at the end of each year, all that will be left in the Stanley Fund at the end of 20 years will be Megan's deposits $= (20)(5000) = 100,000$.

The interest earned in the first year is 0 since the first contribution does not occur until the end of the first year. The interest earned in the second year is $(5000)(0.06) = 300$. This is moved to the Hahn Fund at the end of the second year. The interest earned in the third year is $(2)(5000)(0.06) = 600$. This is moved to the Hahn Fund at the end of the third year. This pattern continues until $(19)(5000)(0.06) = 19(300)$ is moved to the Hahn Fund at the end of the 20th year.

The amount in the Hahn Fund at the end of 20 years is:

$$\left(300a_{\overline{19}|0.05} + \left(\frac{300}{0.05}\right)\left(a_{\overline{19}|0.05} - 19(1.05)^{-19}\right)\right)(1.05)^{19} =$$

$$\left(300\left[\frac{1-(1.05)^{-19}}{0.05}\right] + \left(\frac{300}{0.05}\right)\left(\left[\frac{1-(1.05)^{-19}}{0.05}\right] - 19(1.05)^{-19}\right)\right)(1.05)^{19} = 78,395.72$$

$$Total = 100,000 + 78,395.72 = 178,395.72$$

Ben is the beneficiary of a trust fund. He (or his heirs) will receive a monthly perpetuity from the trust fund.

The monthly payments are 1000 at the end of each month for the first 10 years. Thereafter the payments are each 10 higher than the previous payment. In other words, Ben (or his heirs) will receive 120 payments of 1000 followed by payments of 1010, 1020, 1030, etc.

Calculate the present value of the payments from the trust fund using an interest rate of 12% compounded monthly.

Solution:

You have to split these up into two parts. There are numerous ways to do this. This solution will be 120 payments of 1000. Then beginning with the 121 payment of 1010 it is an arithmetically increasing perpetuity.

$$PV = 1000a_{\overline{120}|} + v^{120}\left[\frac{1010}{i} + \frac{10}{i^2}\right]$$

where the interest rate use is the monthly effective interest rate of $\frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$

$$PV = 1000\left(\frac{1-(1.01)^{-120}}{0.01}\right) + (1.01)^{-120}\left[\frac{1010}{0.01} + \frac{10}{(0.01)^2}\right] = 130,602.47$$

Tom's parents are paying him an annuity for the next 10 years. The annuity makes quarterly payments at the start of each quarter. The first payment is 1000. The second payment is 1500. The third payment is 2000. Each payment continues to increase in the same pattern with each payment being 500 greater than the prior payment.

Calculate the present value of Tom's payments at an interest rate of 10% compounded quarterly.

Solution:

$$PV = \left[1000a_{\overline{40|}} + \left(\frac{500}{0.025}\right)\left(a_{\overline{40|}} - 40(1+i)^{-40}\right)\right](1+i)$$

$$= \left[1000\left(\frac{1-(1.025)^{-40}}{0.025}\right) + \left(\frac{500}{0.025}\right)\left(\left(\frac{1-(1.025)^{-40}}{0.025}\right) - 40(1.025)^{-40}\right)\right](1.025)$$

$$= 234{,}944.12$$

We need the quarterly effective interest rate since payments are quarterly.

$$\frac{i^{(4)}}{4} = \frac{0.10}{4} = 0.025$$

We multiply by (1+i) at the end because the payments are at the beginning of the quarter.