Topic: Non-Level Annuities, Geometric Series

Christopher has won the lottery. He has the choice of the following three options:

a. A lump sum of 10,000,000

b. A perpetuity due of with annual payments of 740,740.74.

c. An annuity with 20 payments at the beginning of each year for 20 years. The first payment is P. The second payment is $P(1.2)$. The third payment is $P(1.2)^2$. The payments continue it the same pattern with each payment being 120% of the prior payment.

All payments have the same present value at an annual interest rate of i.

Determine P.

Solution:

$$10,000,000 = \frac{740,740.74}{i}(1+i)$$

$$10,000,000i = 740,740.74 + 740,740.74i$$

$$9,259,259i = 740,740.74 \quad ==> \quad i = 0.08$$

$$v = (1.08)^{-1}$$

$$10,000,000 = \frac{P - P(1.2)^{20}v^{20}}{1-1.2v}$$

$$= P\left(\frac{1-(1.2)^{20}(1.08)^{-20}}{1-(1.2)(1.08)^{-1}}\right)$$

$$10,000,000 = 65.027P \quad ==> \quad P = 153,781.40$$

The Crawford Family Trust fund will pay Emily monthly payments at the end of each month for 5 years. The first payment is 1000. Each payment after that payment is 102% of the prior payment.

In other words, the first payment will be 1000. The second payment will be $1000(1.02)^1$. The third payment will be $1000(1.02)^2$, etc.

Calculate the present value of these payments at an interest rate of 6% compounded monthly.

Solution:

$$\text{Payments are } 1000,\ 1000(1.02),\ 1000(1.02)^2,\ \ldots,\ 1000(1.02)^{59}$$

$$PV = 1000v + 1000(1.02)v^2 + \ldots + (1000)(1.02)^{59}v^{60}$$

$$\boxed{\text{We need } \frac{i^{(12)}}{12} \text{ which is equal to } \frac{0.06}{12} = 0.005.}$$

$$PV = PV = 1000(1.005)^{-1} + 1000(1.02)(1.005)^{-2} + \ldots + (1000)(1.02)^{59}(1.005)^{-60}$$

$$= \frac{1000(1.005)^{-1} - (1000)(1.02)^{60}(1.005)^{-61}}{1-(1.02)(1.005)^{-1}} = 95{,}497.67$$

Xue is receiving a scholarship from Purdue that pays monthly payments at the beginning of each month for 48 months. The first payment is 1000. The second payment is (1000)(1.01). The third payment is $(1000)(1.01)^2$. The payments continue in the same pattern with each payment being 1.01 times the prior payment.

Using an interest rate of 6% compounded monthly, calculate the present value of Xue's scholarship.

Solution:

$$PV = 1000 + 1000(1.01)(1.005)^{-1} + \ldots + 1000(1.01)^{47}(1.005)^{-47}$$

$$= \frac{1000 - 1000(1.01)^{48}(1.005)^{-48}}{1 - (1.01)(1.005)^{-1}} = 54,065.10$$

Tomas is receiving an annuity due with monthly payments for the next five years. The first payment is 500 at the start of the first month. The second payment is $500(1.08)$ at the beginning of the second month. The third payment is $500(1.08)^2$ at the beginning of the third month. The payments continue to increase in the same pattern.

Calculate the present value of this annuity using an interest rate of 12% compounded monthly.

Solution:

Since payments ar monthly, we need $\frac{i^{(12)}}{12}$ which is $\frac{0.12}{12} = 0.01$

$$PV = 500 + 500(1.08)(1.01)^{-1} + ... + 500(1.08)^{59}(1.01)^{-59}$$

$$= \frac{500 - 500(1.08)^{60}(1.01)^{-60}}{1 - (1.08)(1.01)^{-1}} = 394,887.72$$

Chrissy is the beneficiary of the Maple Trust. The Maple Trust will make payments to Chrissy at the beginning of each year for the next 25 years. The first payment is 1000. The second payment is $(1000)(1.07)^1$. The third payment is $(1000)(1.07)^2$. The payments continue to increase with each payment being 107% of the prior payment.

Calculate the present value of the payments at an annual effective interest rate of 6%.

Solution:

$$PV = 1000 + 1000(1.07)(1.06)^{-1} + ... + 1000(1.07)^{24}(1.06)^{-24}$$

$$= \frac{1000 - 1000(1.07)^{25}(1.06)^{-25}}{1-(1.07)(1.06)^{-1}} = 28,045.94$$

Evan was awarded a scholarship upon his entrance to Purdue. The scholarship makes quarterly payments for four years at the beginning of each quarter. The first payment was 1200. The second payment was $1200(1.1)$. The third payment was $1200(1.1)^2$. The payments continue to increase with each payment being 110% of the prior payment.

Evan takes each payment and invests the payment into an account that earns an annual effective interest rate of 8%.

Determine the amount that Evan will have at the end of four years.

Solution:

This is an annuity with payments in a geometric sequence. The first payment is 1200 and we want the value at the end of 16 periods so the value is $1200(1+i)^{16}$. The value at the end of the second payment is $1200(1.1)(1+i)^{15}$. The payments continue until the 16th payment is made at the end of the 15 quarter and the value one quarter later is $1200(1.1)^{15}(1+i)^{1}$.

$$AV = 1200(1+i)^{16} + 1200(1.1)(1+i)^{15} + ... + 1200(1.1)^{15}(1+i)^{1}$$

$$= \frac{FirstTerm - NextTermAfterLast}{1 - ratio} = \frac{1200(1+i)^{16} - 1200(1.1)^{16}(1+i)^{0}}{1-(1.1)(1+i)^{-1}}$$

Since payments are quarterly, we need the quarterly effective interest rate. We are given the annual effective interest rate.

$$\frac{i^{(4)}}{4} = (1.08)^{0.25} - 1 = 0.019426547$$

$$PV = \frac{1200(1.019426547)^{16} - 1200(1.1)^{16}(1.019426547)^{0}}{1-(1.1)(1.019426547)^{-1}} = 49{,}107.77$$