Topic: Spot and Forward Rates

You are given the following spot interest rate curve:

t	r_t
0.25	0.030
0.50	0.035
0.75	0.039
1.00	0.042
1.25	0.045
1.50	0.047
1.75	0.049
2.00	0.050

Using the above spot interest rates, calculate the present value of an annuity immediate with three semi-annual payments. The first payment is 1000 in six months. The second payment is 2000 in one year. The final payment is 3000 in 18 months.

Solution:

$$PV = 1000(1+r_{0.5})^{-0.5} + 2000(1+r_1)^{-1} + 1000(1+r_{1.5})^{-1.5}$$

$$= 1000(1.035)^{-0.5} + 2000(1.042)^{-1} + 1000(1.047)^{-1.5}$$

$$= 5702.61$$

You are given the following spot interest rate curve:

t	r_t
0.5	0.020
1.0	0.026
1.5	0.031
2.0	0.035
2.5	0.039
3.0	0.042
3.5	0.045
4.0	0.048
4.5	0.051
5.0	0.054

Use the spot interest rates to calculate the accumulated value of an annuity due with payments of 24,000 at the beginning of each year for 3 years.

Solution:

First, we find the present value, then we find the accumulated value.

$$PV = 24,000(1+(1+r_1)^{-1}+(1+r_2)^{-2})$$

$$= 24,000(1+(1.026)^{-1}+(1.035)^{-2})$$

$$= 69,796.07$$

$$AV = PV(1+r_3)^3 = 69,796.07(1.042)^3 = 78,964.91$$

You are given the following spot interest rate curve:

t	r_t
0.5	0.020
1.0	0.026
1.5	0.031
2.0	0.035
2.5	0.039
3.0	0.042
3.5	0.045
4.0	0.048
4.5	0.051
5.0	0.054

Use the spot interest rates to calculate the price of two year par value bond with a maturity value of 10,000. The bond pays semi-annual coupon at a rate of 8% convertible semi-annually.

Solution:

$$\text{Semi-annual Coupon } = Fr = 10,000\left(\frac{0.08}{2}\right) = 400$$

$$Price = PV = 400(1+r_{0.5})^{-0.5} + 400(1+r_1)^{-1} + 400(1+r_{1.5})^{-1.5} + 10,400(1+r_2)^{-2}$$

$$= 400(1.02)^{-0.5} + 400(1.026)^{-1} + 400(1.031)^{-1.5} + 10,400(1.035)^{-2}$$

$$= 10,876.53$$

You are given the following three bonds:

a. A one year bond which sells for 990 and has a maturity value of 1000 and annual coupons of 42.

b. A two year bond with a maturity value of 50,000 and annual coupons of 10,000. The price of this bond is 61,000.

c. A three year zero coupon bond with a maturity value of 100,000 which sells for 77,000.

Use bootstrapping to find $f_{[1,3]}$.

Solution:

First, we need to find the spot rates using bootstrapping. Then, using the spot rates, we will find the forward rate.

$$Price\ of\ A\ = 990 = \frac{1042}{1+r_1} \quad ==> \quad r_1 = \frac{1042}{990} - 1 = 0.052525253$$

$$Price\ of\ B\ = 61,000 = \frac{10,000}{1+r_1} + \frac{60,000}{(1+r_2)^2}$$

$$\frac{60,000}{(1+r_2)^2} = 61,000 - \frac{10,000}{1.052525253} = 51,499.04 \Rightarrow r_2 = \left(\frac{60,000}{51,499.04}\right)^{\frac{1}{2}} - 1 = 0.079384202$$

$$Price\ of\ C\ = 77,000 = \frac{10,000}{(1+r_3)^3} \quad ==> \quad r_3 = \left(\frac{100,000}{77,000}\right)^{\frac{1}{3}} - 1 = 0.091029328$$

$$(1+r_1)(1+f_{[1,3]})^2 = (1+r_3)^3 \quad ==> \quad f_{[1,3]} = \left(\frac{(1+r_3)^3}{1+r_1}\right)^{\frac{1}{2}} - 1$$

$$f_{[1,3]} = \left(\frac{(1.091029328)^3}{1.052525253}\right)^{\frac{1}{2}} - 1 = 0.110806405$$

Luke can buy the following two bonds:

a. Bond A is a one year bond with a price of 1020. The bond has annual coupons of 90 and a maturity value of 1000.

b. Bond B is a two year bond with a maturity value of 10,000. The bond has annual coupons of 700. The price of the bond is 9900.

Based on these two bonds, determine $f_{[1,2]}$.

(Hint: Use bootstrapping to find the spot rates and then find the forward rate.)

Solution:

First, we need to find the spot rates using bootstrapping. Then, using the spot rates, we will find the forward rate.

$$Price\ of\ A = 1020 = \frac{1090}{1+r_1} \Rightarrow r_1 = \frac{1090}{1020} - 1 = 0.068627451$$

$$Price\ of\ B = 9900 = \frac{700}{1+r_1} + \frac{10{,}700}{(1+r_2)^2}$$

$$\frac{10{,}700}{(1+r_2)^2} = 9900 - \frac{700}{1.068627451} = 9244.95 \Rightarrow r_2 = \left(\frac{10{,}700}{9244.95}\right)^{0.5} - 1 = 0.075819739$$

$$f_{[1,2]} = \frac{(1+r_2)^2}{1+r_1} - 1 = \frac{(1.075819739)^2}{1.068627451} - 1 = 0.083060434$$

You are given the following spot interest rates:

t	r_t	t	r_t
0.5	5.00%	3.0	6.60%
1.0	5.40%	3.5	6.85%
1.5	5.75%	4.0	7.05%
2.0	6.05%	4.5	7.20%
2.5	6.35%	5.0	7.30%

Grigor is receiving an annuity immediate with two annual payments of 10,000.

Using the spot interest rates, you determine the present value of the annuity. You then determine the equivalent annual yield rate on the annuity.

What was the annual yield rate that you determined?

Solution:

$$PV = 10{,}000(1+r_1)^{-1} + 10{,}000(1+r_2)^{-2} = 10{,}000(1.054)^{-1} + 10{,}000(1.0605)^{-2}$$

$$PV = 18{,}379.24018$$

We want to find i , the annual yield rate.

$$18{,}379.24018 = 10{,}000\, a_{\overline{2|}}$$

Use your calculator to find i :

$$\boxed{N} \leftarrow 2\ ;\ \boxed{PV} \leftarrow 18{,}379.24018\ ;\ \boxed{PMT} \leftarrow -10{,}000$$

$$\boxed{CPT}\ \boxed{I/Y} \leftarrow 5.824018766\%$$

You are given the following spot interest rates:

t	r_t	t	r_t
0.5	4.00%	3.0	5.85%
1.0	4.50%	3.5	6.20%
1.5	4.95%	4.0	6.35%
2.0	5.30%	4.5	6.45%
2.5	5.60%	5.0	6.50%

Determine the price of a 2 year bond with semi-annual coupons of 200 and a maturity value of 3000.

Solution:

$$P = PV = (200)(1.04)^{-0.5} + (200)(1.045)^{-1} + (200)(1.0495)^{-1.5} + 3200(1.053)^{-2}$$

$$= 3459.50$$

You are given the following spot interest rates:

t	r_t	t	r_t
0.5	4.00%	3.0	5.85%
1.0	4.50%	3.5	6.20%
1.5	4.95%	4.0	6.35%
2.0	5.30%	4.5	6.45%
2.5	5.60%	5.0	6.50%

Tianjian invested 10,000 today and another 10,000 at the end of two years.

How much will Tianjian have at the end of four years?

Solution:

First, find the present value using the spot rates.

$$PV = 10,000 + 10,000(1.053)^{-2} = 19,018.68582$$

Then $AV = PV(1.0635)^4 = 24,329.35$

You are given the following spot interest rate curve:

t	r_t
0.5	1.25%
1.0	2.00%
1.5	2.75%
2.0	3.20%
2.5	3.60%
3.0	4.00%

Using these spot interest rates, determine the price of a 2 year bond that matures for 100,000 and has semi-annual coupons of 5000.

Solutions:

$$PV = 5000(1.0125)^{-0.5} + 5000(1.02)^{-1} + 5000(1.0275)^{-1.5} + 105,000(1.032)^{-2}$$

$$= 113,260.95$$

You are given the following spot interest rates:

t	r_t	t	r_t
0.5	3.00%	3.0	4.85%
1.0	3.50%	3.5	5.20%
1.5	3.95%	4.0	5.35%
2.0	4.30%	4.5	5.45%
2.5	4.60%	5.0	5.50%

Determine the present value of an annuity immediate with semi-annual payments for two years. The payments increase each payment. The first payment is 100. The second payment is 200. The third payment is 400. The final payment is 800.

Solution:

$$PV = 100(1.03)^{-0.5} + 200(1.035)^{-1} + 400(1.0395)^{-1.5} + 800(1.043)^{-2}$$

$$= 1404.58$$

You are given the following spot interest rates:

t	r_t	t	r_t
0.5	3.00%	3.0	4.85%
1.0	3.50%	3.5	5.20%
1.5	3.95%	4.0	5.35%
2.0	4.30%	4.5	5.45%
2.5	4.60%	5.0	5.50%

Determine the accumulated value of an annuity due that pays 1000 at the beginning of each year for 3 years.

Solution:

First, we find the present value and then find the accumulated value.

$$PV = 1000(1 + (1.035)^{-1} + (1.043)^{-2}) = 2885.43$$

$$AV = PV(1.0485)^3 = 3325.95$$

You are given the following two bonds:

a. Bond 1 is a one year zero coupon bond with a price of 9400 and a maturity value of 10,000.

b. Bond 2 is a two year bond with annual coupons of 300 and a maturity value of 2000. This bond sells for an annual yield of 8%.

You are also given that the three year spot interest rate is 9%.

Determine the price of a three year bond with annual coupons of 800 and a maturity value of 3000.

Solution:

Using Bond 1

$$9400 = \frac{10,000}{1+r_1} \Longrightarrow r_1 = \frac{10,000}{9400} - 1 = 0.063829787$$

Using Bond 2

$$\frac{300}{1+r_1} + \frac{2300}{(1+r_2)^2} = \frac{300}{1.08} + \frac{2300}{(1.08)^2} \Longrightarrow \frac{300}{1.063829787} + \frac{2300}{(1+r_2)^2} = 2249.66$$

$$r_2 = \frac{2300}{2249.66 - \dfrac{300}{1.063829787}} - 1 = 0.081158118$$

$$\text{Price} = \frac{800}{1+r_1} + \frac{800}{(1+r_2)^2} + \frac{3800}{(1+r_3)^3} = \frac{800}{1.063829787} + \frac{800}{(1.081158118)^2} + \frac{3800}{(1.09)^3} = 4370.70$$

You are given the following spot interest rates:

Time (t)	r_t
0.5	2.1%
1.0	2.3%
1.5	2.6%
2.0	3.0%
2.5	3.3%
3.0	3.7%
3.5	4.2%
4.0	4.8%
4.5	5.5%
5.0	6.0%

Megan purchases a two year par value bond with semi-annual coupons at a rate of 8% convertible semi-annually. The par value of the bond is 10,000.

Calculate the price of the bond.

Solution:

$$Coupon = (10,000)(0.08/2) = 400$$

$$P = \frac{400}{(1.021)^{0.5}} + \frac{400}{(1.023)^{1}} + \frac{400}{(1.026)^{1.5}} + \frac{10,400}{(1.03)^{2}} = 10,974.76$$

The following three bonds are priced using the same spot interest rates:

a. Bond A is a one year bond with a maturity value of 1000, annual coupons of 70, and a price of 1005.

b. Bond B is a two year bond with annual coupons of 200 and a maturity value of 1000. It sells to yield an annual effective interest rate of 8%.

c. Bond C is a three year bond with annual coupons of 270. The maturity value and the price of the bond are 3000.

Kristin has a three year annuity immediate with annual payments of 1000. Using the same spot interest rates, determine the accumulated value of Kristin's annuity.

Solution:

First we need to find the spot rates using bootstrapping. Then, using the spot rates, we will find the present value of the annuity. Finally, we will find the accumulated value of the annuity.

$$\frac{1070}{1+r_1} = 1005 \implies r_1 = \frac{1070}{1005} - 1 = 0.06467662$$

$$\frac{200}{1+r_1} + \frac{1200}{(1+r_2)^2} = \text{Price of B} = 200a_{\overline{2}|} + 1000v^2 = 200\left(\frac{1-(1.08)^{-2}}{0.08}\right) + 1000(1.08)^{-2} = 1213.99$$

$$\frac{1200}{(1+r_2)^2} = 1213.99 - \frac{200}{1.06467662} = 1026.14 \implies r_2 = \left(\frac{1200}{1026.14}\right)^{0.5} - 1 = 0.081402$$

$$\frac{270}{1+r_1} + \frac{270}{(1+r_2)^2} + \frac{3270}{(1+r_3)^3} = 3000 \implies \frac{3270}{(1+r_3)^3} = 3000 - \frac{270}{1.06467662} - \frac{270}{(1.081402)^2} = 2515.5202$$

$$r_3 = \left(\frac{3270}{2515.5202}\right)^{1/3} - 1 = 0.091373$$

$$PV = \frac{1000}{1+r_1} + \frac{1000}{(1+r_2)^2} + \frac{1000}{(1+r_3)^3} = \frac{1000}{1.06467662} + \frac{1000}{(1.081402)^2} + \frac{1000}{(1.091373)^3} = 2563.64$$

$$AV = PV(1+r_3)^3 = (2563.64)(1.091373)^3 = 3332.55$$

You are given:

a. $f_{[0,1]} = 0.05$

b. $f_{[1,2]} = 0.06$

c. $f_{[2,3]} = 0.07$

d. $f_{[3,4]} = 0.08$

Calculate the price of a zero coupon bond that matures for 100,000 at the end of four years.

Solution:

$$P = \frac{100,000}{(1.05)(1.06)(1.07)(1.08)} = 77,749.45$$

You can buy the following three bonds:

i. A six month zero coupon bond with a maturity value of 1000 and a price of 970.

ii. A one year bond with semi-annual coupons of 100 and a maturity value of 1000. The price of the bond is 1130.

iii. A bond that matures in 18 months with semi-annual coupons of 400 and a maturity value of 800. The price of this bond is 1845.

Determine the spot rate for 18 months.

Solution:

Using Bond i:

$$970(1+r_{0.5})^{0.5}=1000 \Longrightarrow r_{0.5}=0.062812201$$

Using Bond ii:

$$1130=100(1+r_{0.5})^{-0.5}+1100(1+r_1)^{-1} \Longrightarrow r_1=\frac{1100}{1130-100(1.062812201)^{-0.5}}-1=0.064859632$$

Using Bond iii:

$$1845=400(1+r_{0.5})^{-0.5}+400(1+r_1)^{-1}+1200(1+r_{1.5})^{-1.5}$$

$$1845=400(1.062812201)^{-0.5}+400(1.064859632)^{-1}+1200(1+r_{1.5})^{-1.5}$$

$$r_{1.5}=\left(\frac{1200}{1845-400(1.062812201)^{-0.5}-400(1.064859632)^{-1}}\right)^{1/1.5}-1=0.071864$$

You are given the following three bonds:

a. A one year bond with annual coupons of 100 and a maturity value of 1000. The bond has a price of 1050.

b. A two year bond with annual coupons of 80 and maturity value of 1000. The bond has a price of 990.

c. A three year bond with annual coupons of 200 and a maturity value of 800. The price of this bond is 1065.

Use bootstrapping to determine the three year spot interest rate.

Solution:

Using one year bond:

$$1050 = 1100(1+r_1)^{-1} \implies r_1 = \frac{1100}{1050} - 1 = 0.047619$$

Using two year bond:

$$990 = 80(1+r_1)^{-1} + 1080(1+r_2)^{-2} = 80(1.047619)^{-1} + 1080(1+r_2)^{-2}$$

$$\implies r_2 = \left(\frac{1080}{990 - 80(1.047619)^{-1}}\right)^{0.5} - 1 = 0.087239$$

Using three year bond:

$$1065 = 200(1+r_1)^{-1} + 200(1+r_2)^{-2} + 1000(1+r_3)^{-3} =$$

$$200(1.047619)^{-1} + 200(1.087239)^{-2} + 1000(1+r_3)^{-3}$$

$$\implies r_3 = \left(\frac{1000}{1065 - 200(1.047619)^{-1} - 200(1.087239)^{-2}}\right)^{1/3} - 1 = 0.123633$$