Chapter 2 – Past Test and Quiz Problems – Complete and Curtate Expectation of Life

(10 points)

You are given that ${}_t p_{90} = 1 - 0.04t^2$ for $0 \le t \le 5$.

Calculate the $Var[T_{90}] - Var[K_{90}]$.

Solution:

$$Var[T_{90}] = E[T_{90}{}^2] - \left(E[T_{90}]\right)^2 = 12.5 - (3.33333)^2 = 1.38889$$

$$E[T_{90}] = \int_0^5 {}_t p_{90} \cdot dt = \int_0^5 (1 - 0.04t^2) \cdot dt = \left[t - \frac{0.04t^3}{3} \right]_0^5 = \left[5 - \frac{0.04(5)^3}{3} \right] = 3.33333$$

$$E[T_{90}{}^2] = 2\int_0^5 t \cdot {}_t p_{90} \cdot dt = 2\int_0^5 (t)(1 - 0.04t^2) \cdot dt = 2\left[\frac{t^2}{2} - \frac{0.04t^4}{4} \right]_0^5 = 2\left[\frac{(5)^2}{2} - \frac{0.04(5)^4}{4} \right] = 12.5$$

$$Var[K_{90}] = E[K_{90}{}^2] - \left(E[K_{90}]\right)^2 = 9.2 - (2.8)^2 = 1.36$$

$$E[K_{90}] = \sum_{k=1}^{5} {}_t p_{90} = \sum_{k=1}^{5} (1 - 0.04t^2) =$$

$$(1 - 0.04(1)^2) + (1 - 0.04(2)^2) + (1 - 0.04(3)^2) + (1 - 0.04(4)^2) + (1 - 0.04(5)^2) = 2.8$$

$$E[K_{90}{}^2] = 2\sum_{k=1}^{5} t \cdot {}_t p_{90} - \sum_{k=1}^{5} {}_t p_{90} = \left[2\sum_{k=1}^{5} t(1 - 0.04t^2) \right] - 2.8$$

$$= (1)(1 - 0.04(1)^2) + (2)(1 - 0.04(2)^2) + (3)(1 - 0.04(3)^2) + (4)(1 - 0.04(4)^2) + (5)(1 - 0.04(5)^2) - 2.8$$

$$= 9.2$$

$$Var[T_{90}] - Var[K_{90}] = 1.38889 - 1.36 = \boxed{0.02889}$$

(4 points)

You are given that ${}_t p_x = 1 - \frac{t^3}{n^3}$ for $0 \le t \le n$.

You are also given that $\overset{\circ}{e}_x = 4.5$

Calculate n .

Solution:

$$\overset{\circ}{e}_x = \int_0^n {}_t p_x dt = \int_0^n \left(1 - \frac{t^3}{n^3}\right) dt = \left[t - \frac{t^4}{4n^3}\right]_0^n = n - \frac{n}{4} = 0.75n = 4.5 \Longrightarrow n = \frac{4.5}{0.75} = 6$$

(6 points)

You are given that ${}_tq_{90} = \frac{t^3}{1000}$.

Calculate the $Var[T_{90}]$.

Solution:

${}_tp_{90} = 1 - {}_tq_{90} = 1 - \frac{t^3}{1000}$

${}_tp_{90} = \frac{1000-t^3}{1000}$

$Var[T_x] = E[T_x^2] - E[T_x]^2$

$$\mathring{e}_{90} = E[T_{90}] = \frac{1}{1000}\int_0^{10}(1000 - t^3)dt$$

$$E[T_{90}] = \frac{1}{1000}[1000t - \frac{t^4}{4}]_0^{10} = 7.5$$

$$E[T_{90}^2] = (2)\frac{1}{1000}\int_0^{10} t(1000 - t^3)dt$$

$$E[T_{90}^2] = \frac{1}{500}[500t^2 - \frac{t^5}{5}]_0^{10} = 60$$

$Var[T_{90}] = 60 - 7.5^2 = 3.75$

Note that the integral is from 0 to 10, because the q-value becomes greater than 1, and therefore invalid, for any integer greater than 10. Therefore you can treat $(\omega - x)$ as 10.

(6 points)

You are given than ${}_tq_{95} = \frac{t^3}{125}$.

Calculate the $Var\left(K_{95}\right)$.

Solution:

${}_tp_{95} = 1 - {}_tq_{95} = 1 - \frac{t^3}{125}$

${}_tp_{95} = \frac{125 - t^3}{125}$

$Var[K_x] = E[K_x^2] - E[K_x]^2$

$$E[K_{95}] = \sum_{K=1}^{5} {}_kp_x$$

$$= \frac{125-1^3}{125} + \frac{125-2^3}{125} + \frac{125-3^3}{125} + \frac{125-4^3}{125} + \frac{125-5^3}{125}$$

$$= 3.2$$

$$E[K_{95}^2] = 2\sum_{k=1}^{5} k \cdot {}_kp_x - E[K_{95}]$$

$$= 2[(1)\frac{125-1^3}{125} + (2)\frac{125-2^3}{125} + (3)\frac{125-3^3}{125} + (4)\frac{125-4^3}{125}] - 3.2$$

$$E[K_{95}^2] = 11.136$$

$Var[K_{95}] = 11.136 - 3.2^2 = 0.896$

Note that the summation is from 1 to 5, because the p-value goes negative, and therefore invalid, for any integer greater than 5. Therefore you can treat $(\omega - x)$ as 5.

(6 points)

You are given that ${}_t p_{95} = 1 - 0.04t^2$ for $0 \le t \le 5$

Calculate $Var[K_{95}]$.

Solution:

$$Var[K_{95}] = E[K_{95}{}^2] - (E[K_{95}])^2$$

$$E[K_{95}] = \sum_{k=1}^{4} {}_k p_{95} = \sum_{k=1}^{4} (1 - 0.04k^2)$$

$$= 1 - 0.04(1^2) + 1 - 0.04(2^2) + 1 - 0.04(3^2) + 1 - 0.04(4^2) = 2.8$$

$$E[K_{95}{}^2] = 2\sum_{k=1}^{4} k \cdot {}_k p_{95} \ - E[K_x] = 2\sum_{k=1}^{4} k(1 - 0.04k^2) \ - 2.8$$

$$= 2\left\{1\left[1 - 0.04(1^2)\right] + 2\left[1 - 0.04(2^2)\right] + 3\left[1 - 0.04(3^2)\right] + 4\left[1 - 0.04(4^2)\right]\right\} - 2.8 = 9.2$$

$$Var[K_{95}] = 9.2 - (2.8)^2 = 1.36$$

(6 points)

You are given that ${}_tq_{75} = \dfrac{t^2+t}{240}$ for $0 \le t \le 15$.

Calculate $Var[T_{75}]$

Solution:

$$Var[T_{75}] = E[T_{75}{}^2] - \left(E[T_{75}]\right)^2$$

$$E[T_{75}] = \int_0^{15} {}_tp_{75} \cdot dt = \int_0^{15} (1 - {}_tq_{75}) \cdot dt = \int_0^{15} \left[1 - \frac{t^2+t}{240}\right] \cdot dt$$

$$= \left[t - \frac{t^3}{720} - \frac{t^2}{480}\right]_0^{15} = 15 - \frac{15^3}{720} - \frac{15^2}{480} = 9.84375$$

$$E[T_{75}{}^2] = 2\int_0^{15} t \cdot {}_tp_{75} \cdot dt = 2\int_0^{15} t(1 - {}_tq_{75}) \cdot dt = 2\int_0^{15} \left[t\left(1 - \frac{t^2+t}{240}\right)\right] \cdot dt$$

$$= 2\left[\frac{t^2}{2} - \frac{t^4}{960} - \frac{t^3}{720}\right]_0^{15} = 2\left[\frac{15^2}{2} - \frac{15^4}{960} - \frac{15^3}{720}\right] = 110.15625$$

$$Var[T_{75}] = 110.15625 - (9.84375)^2 = 13.25684$$