1. You are given:

$$F_0(t) = 1 - \left(1 - \frac{t}{125}\right)^{\frac{1}{5}}, 0 \le t \le 125$$

Calculate:

a. $S_0(t)$

$$S_0(t) = 1 - F_0(t) = \left(1 - \frac{t}{125}\right)^{\frac{1}{5}}, \quad 0 \le t \le 125$$

b. $\Pr[T_0 \le t]$

$$\Pr\left[T_0 \le t\right] = F_0(t) = 1 - \left(1 - \frac{t}{125}\right)^{\frac{1}{5}}, \quad 0 \le t \le 125$$

c. $\Pr[T_0 > t]$

$$\Pr\left[T_0 > t\right] = S_0(t) = \left(1 - \frac{t}{125}\right)^{\frac{1}{5}}, \quad 0 \le t \le 125$$

d. $S_x(t)$

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{\left(1 - \frac{x+t}{125}\right)^{\frac{1}{5}}}{\left(1 - \frac{x}{125}\right)^{\frac{1}{5}}} = \left(\frac{125 - x - t}{125 - x}\right)^{\frac{1}{5}} = \left(1 - \frac{t}{125 - x}\right)^{\frac{1}{5}}, \quad 0 \le t \le 125 - x$$

e. Probability that a newborn will live to age 25.

$$S_0(25) = \left(1 - \frac{25}{125}\right)^{\frac{1}{5}} \approx 0.95635$$

f. Probability that a person age 25 will live to age 75.

$$S_{25}(50) = \left(\frac{125 - 25 - 50}{125 - 25}\right)^{\frac{1}{5}} \approx 0.87055$$

g. Probability that a person age 25 will die between age 50 and age 75.

$$S_{25}(25) - S_{25}(50) = \left(\frac{125 - 25 - 25}{125 - 25}\right)^{\frac{1}{5}} - \left(\frac{125 - 25 - 50}{125 - 25}\right)^{\frac{1}{5}} \approx 0.07354$$

h. ω

$W = \text{Maximum age one person can live} = 125$

i. μ_x

$$\frac{d}{dt}S_0(x)=\frac{1}{5}\cdot\left(1-\frac{t}{125}\right)^{-\frac{4}{5}}\cdot\left(-\frac{1}{125}\right)$$

$$\mu_x=-\frac{\frac{d}{dt}S_0(x)}{S_0(x)}=\frac{\frac{1}{5}\cdot\left(1-\frac{t}{125}\right)^{-\frac{4}{5}}\cdot\frac{1}{125}}{\left(1-\frac{t}{125}\right)^{\frac{1}{5}}}=\frac{1}{625}\cdot\left(1-\frac{t}{125}\right)^{-1}=\frac{1}{5\cdot(125-t)}$$

j. μ_{25}

$$\mu_{25}=\frac{1}{625}\cdot\left(1-\frac{25}{125}\right)^{-1}=0.002$$

k. μ_{100}

$$\mu_{100}=\frac{1}{625}\cdot\left(1-\frac{100}{125}\right)^{-1}=0.008$$

l. ${}_t p_x$

$${}_t p_x=S_x(t)=\left(1-\frac{t}{125-x}\right)^{\frac{1}{5}}$$

m. ${}_{10}p_{50}$

$${}_{10}p_{50}=\left(1-\frac{10}{125-50}\right)^{\frac{1}{5}}\approx 0.97179$$

n. ${}_t q_x$

$${}_t q_x=1-{}_t p_x=1-\left(1-\frac{t}{125-x}\right)^{\frac{1}{5}}$$

o. ${}_{10}q_{50}$

$${}_{10}q_{50}=1-\left(1-\frac{10}{125-50}\right)^{\frac{1}{5}}\approx 0.028214$$

p. ${}_{10}p_{50}+{}_{10}q_{50}$

$${}_{10}p_{50}+{}_{10}q_{50}=1$$

q. p_{50}

$$p_{50}={}_1 p_{50}=\left(1-\frac{1}{125-50}\right)^{\frac{1}{5}}\approx 0.99732$$

r. $_{u|t}q_x$

$$_{u|t}q_x = {}_u p_x - {}_{u+t} p_x = \left(1 - \frac{u}{125 - x}\right)^{\frac{1}{5}} - \left(1 - \frac{u+t}{125 - x}\right)^{\frac{1}{5}}$$

$$= \frac{(125 - x - u)^{\frac{1}{5}} - (125 - x - u - t)^{\frac{1}{5}}}{(125 - x)^{\frac{1}{5}}}$$

s. $f_x(t)$

$$f_x(t) = \frac{d}{dt} F_x(t) = -\frac{1}{5} \cdot \left(1 - \frac{t}{125 - x}\right)^{-\frac{4}{5}} \cdot \left(-\frac{1}{125 - x}\right) = \frac{1}{625 - 5x} \cdot \left(1 - \frac{t}{125 - x}\right)^{-\frac{4}{5}}$$

t. $E[T_x]$

$$E[T_x] = \int_0^{125-x} {}_t p_x \, dt$$

$$= \int_0^{125-x} \left(1 - \frac{t}{125 - x}\right)^{\frac{1}{5}} dt$$

$$= \left[-(125 - x) \cdot \frac{5}{6} \cdot \left(1 - \frac{t}{125 - x}\right)^{\frac{6}{5}} \right]_0^{125-x}$$

$$= 0 - \left[-(125 - x) \cdot \frac{5}{6} \right]$$

$$= \frac{5}{6}(125 - x)$$

u. $\overset{\circ}{e}_x$

$$\overset{\circ}{e}_x = E[T_x] = \frac{5}{6}(125 - x)$$

v. $Var[T_x]$

$$E\left[T_x^{\,2}\right] = 2\int_0^{125-x} t \cdot {}_t p_x \, dt$$

$$= 2\int_0^{125-x} t \cdot \left(1 - \frac{t}{125-x}\right)^{\frac{1}{5}} dt$$

$$= 2\int_0^{125-x} t \cdot d\left[(x-125)\cdot\frac{5}{6}\cdot\left(1-\frac{t}{125-x}\right)^{\frac{6}{5}}\right]$$

$$= \left[2t\cdot(x-125)\cdot\frac{5}{6}\cdot\left(1-\frac{t}{125-x}\right)^{\frac{6}{5}}\right]_0^{125-x} - \int_0^{125-x} 2(x-125)\cdot\frac{5}{6}\cdot\left(1-\frac{t}{125-x}\right)^{\frac{6}{5}} dt$$

$$= [0-0] + (125-x)\cdot\frac{5}{3}\cdot\int_0^{125-x}\left(1-\frac{t}{125-x}\right)^{\frac{6}{5}} dt$$

$$= (125-x)\cdot\frac{5}{3}\cdot\frac{5}{11}\cdot\left(1-\frac{t}{125-x}\right)^{\frac{11}{5}}\cdot\left[-(125-x)\right]\Bigg|_0^{125-x}$$

$$= 0 - \left[-(125-x)^2\cdot\frac{25}{33}\right]$$

$$= \frac{25}{33}\cdot(125-x)^2$$

$$Var\left[T_x\right] = E\left[T_x^{\,2}\right] - \left(E\left[T_x\right]\right)^2$$

$$= (125-x)^2\left(\frac{25}{33} - \frac{25}{36}\right)$$

$$= \frac{25}{396}(125-x)^2$$

w. Standard Deviation of T_{50}

$$\text{StDev}\left[T_{50}\right] = \sqrt{Var\left[T_{50}\right]} = \sqrt{\frac{25}{396}(125-50)^2} = 18.84446$$

x. $E[K_{120}]$

$${}_1p_{120} = \left(1 - \frac{1}{125 - 120}\right)^{\frac{1}{5}} = \left(\frac{4}{5}\right)^{\frac{1}{5}}$$

$${}_2p_{120} = \left(1 - \frac{2}{125 - 120}\right)^{\frac{1}{5}} = \left(\frac{3}{5}\right)^{\frac{1}{5}}$$

$${}_3p_{120} = \left(1 - \frac{3}{125 - 120}\right)^{\frac{1}{5}} = \left(\frac{2}{5}\right)^{\frac{1}{5}}$$

$${}_4p_{120} = \left(1 - \frac{4}{125 - 120}\right)^{\frac{1}{5}} = \left(\frac{1}{5}\right)^{\frac{1}{5}}$$

$${}_5p_{120} = \left(1 - \frac{5}{125 - 120}\right)^{\frac{1}{5}} = 0$$

$$e_{120} = E\left[K_{120}\right] = \sum_{t=1}^{4} {}_tp_{120} = {}_1p_{120} + {}_2p_{120} + {}_3p_{120} + {}_4p_{120} \approx 3.41657$$

y. $Var[K_{120}]$

$$E\left[{K_{120}}^2\right] = 2\sum_{t=1}^{4} t \cdot {}_tp_{120} - e_{120} = 2 \times \left({}_1p_{120} + 2 \times {}_2p_{120} + 3 \times {}_3p_{120} + 4 \times {}_4p_{120}\right) - e_x \approx 12.90122$$

$$Var\left[K_{120}\right] = E\left[{K_{120}}^2\right] - \left(E\left[K_{120}\right]\right)^2 \approx 1.22830$$

2. You are given that mortality follows Gompertz Law with B = 0.00027 and c = 1.1. Calculate:
   a. μ_x

      $\mu_x = Bc^x = (0.00027)(1.1)^x$
   b. μ_{25}

      $\mu_{25} = (0.00027)(1.1^{25}) = 0.0029254$
   c. μ_{100}

      $\mu_{100} = (0.00027)(1.1^{100}) = 3.72077$
   d. $S_0(t)$

      $S_0(t) = {}_t p_0 = e^{-\frac{B}{\ln c}c^0(c^t-1)} = e^{-\left(\frac{0.00027}{\ln(1.1)}\right)(1)(1.1^t-1)}$
   e. $\Pr[T_0 \le t]$

      $\Pr[T_0 \le t] = 1 - S_0(t) = 1 - e^{-\left(\frac{0.00027}{\ln(1.1)}\right)(1.1^t-1)}$
   f. $\Pr[T_0 > t]$

      $\Pr[T_0 > t] = S_0(t) = e^{-\left(\frac{0.00027}{\ln(1.1)}\right)(1.1^t-1)}$
   g. $S_x(t)$

      $S_x(t) = {}_t p_x = e^{-\left(\frac{B}{\ln c}\right)(c^x)(c^t-1)} = e^{-\left(\frac{0.00027}{\ln(1.1)}\right)(1.1^x)(1.1^t-1)}$
   h. Probability that a newborn will live to age 25.

      ${}_{25}p_0 = e^{-\frac{0.00027}{\ln(1.1)}(1.1^0)(1.1^{25}-1)} = 0.97252$
   i. Probability that a person age 25 will live to age 75.

      ${}_{50}p_{25} = e^{-\frac{0.00027}{\ln 1.1}(1.1^{25})(1.1^{50}-1)} = 0.028088$
   j. Probability that a person age 25 will die between age 50 and age 75.

      ${}_{25}p_{25} - {}_{50}p_{25} = e^{-\frac{0.00027}{\ln(1.1)}(1.1^{25})(1.1^{25}-1)} - e^{-\frac{0.00027}{\ln(1.1)}(1.1^{25})(1.1^{50}-1)} = 0.71135$
   k. ω

      $\omega = \text{Maximum Age} = \infty$
   l. ${}_t p_x$

      ${}_t p_x = e^{-\frac{B}{\ln c}(c^x)(c^t-1)} = e^{-\frac{0.00027}{\ln(1.1)}(1.1^x)(1.1^t-1)}$
   m. ${}_{10}p_{50}$

      ${}_{10}p_{50} = e^{-\frac{0.00027}{\ln(1.1)}(1.1^{50})(1.1^{10}-1)} = 0.58860$

n. ${}_t q_x$

$${}_t q_x = 1 - {}_t p_x = 1 - e^{-\frac{0.00027}{\ln(1.1)}\left(1.1^x\right)\left(1.1^t - 1\right)}$$

o. ${}_{10}q_{50}$

$${}_{10}q_{50} = 1 - e^{-\frac{0.00027}{\ln(1.1)}\left(1.1^{50}\right)\left(1.1^{10} - 1\right)} = 0.41140$$

p. ${}_{10}p_{50} + {}_{10}q_{50}$

$${}_{10}p_{50} + {}_{10}q_{50} = 1$$

q. p_{50}

$$p_{50} = {}_1 p_{50} = e^{-\frac{0.00027}{\ln(1.1)}\left(1.1^{50}\right)\left(1.1^{1} - 1\right)} = 0.96729$$

r. ${}_{u|t}q_x$

$${}_{u|t}q_x = {}_u p_x - {}_{u+t} p_x = e^{-\frac{B}{\ln c}\left(c^x\right)\left(c^u - 1\right)} - e^{-\frac{B}{\ln c}\left(c^x\right)\left(c^{u+t} - 1\right)} = e^{-\frac{0.00027}{\ln(1.1)}\left(1.1^x\right)\left(1.1^u - 1\right)} - e^{-\frac{0.00027}{\ln(1.1)}\left(1.1^x\right)\left(1.1^{u+t} - 1\right)}$$

s. $f_x(t)$

$$f_x(t) = -\frac{d}{dt}S_x(t) = -\frac{d}{dt}\left(e^{-\frac{B}{\ln c}\left(c^x\right)\left(c^t - 1\right)}\right)$$

$$= -e^{-\frac{B}{\ln c}\left(c^x\right)\left(c^t - 1\right)} \cdot \left(-\frac{B}{\ln c} \cdot c^x\right) \cdot c^t \cdot \ln c$$

$$= e^{-\frac{B}{\ln c}\left(c^x\right)\left(c^t - 1\right)} \cdot Bc^{x+t}$$

$$= 0.00027 \times 1.1^{x+t} \cdot e^{-\frac{0.00027}{\ln(1.1)}\left(1.1^x\right)\left(1.1^t - 1\right)}$$

3. You are given that that $\mu_x = c$ for all $x \geq 0$ where c is a constant. This mortality law is known as a constant force of mortality.

   a. ${}_t p_x$

   $${}_t p_x = e^{-\int_0^t \mu_s ds} = e^{-\int_0^t c \cdot ds} = e^{-\mu_s t}$$

   b. ${}_t q_x$

   $${}_t q_x = 1 - {}_t p_x = 1 - e^{-ct}$$

   c. ω

   $$\omega = \text{Maximum Age} = \infty$$

   d. $\mathring{e}_x$

   $$\mathring{e}_x = \int_0^\infty {}_t p_x \, dt = \int_0^\infty e^{-ct} \, dt = \frac{1}{-c}\left[e^{-ct}\right]_{t=0}^{t=\infty} = \frac{1}{c}$$

   e. $Var[T_x]$

   $$E\left[T_x^{\,2}\right] = 2\int_0^\infty t \cdot {}_t p_x dt = 2\int_0^\infty t \cdot e^{-ct} dt$$

   $$= 2\left[\left[\frac{t \cdot e^{-ct}}{c}\right]_0^\infty - \int_0^\infty \frac{e^{-ct}}{c} dt\right] \Leftarrow\!\!= \text{By Parts}$$

   $$2\left[0 - 0 - \left[\frac{e^{-ct}}{c^2}\right]_0^\infty\right] = \frac{2}{c^2}$$

   $$Var\left[T_x\right] = E\left[T_x^{\,2}\right] - \left(E\left[T_x\right]\right)^2 = \frac{2}{c^2} - \left(\frac{1}{c}\right)^2 = \frac{1}{c^2}$$

   f. e_x

   $$e_x = \sum_{k=1}^{\infty} {}_t p_x = e^{-c} + e^{-2c} + e^{-3c} + \cdots = \frac{e^{-c} - 0}{1 - e^{-c}} = \frac{1}{e^c - 1}$$

   g. ${}_{10}p_{10}$

   Since ${}_t p_x = e^{-ct}$,

   $${}_{10}p_{10} = e^{-10c}$$

   h. ${}_{10}p_{100}$

   $${}_{10}p_{100} = e^{-10c}$$

   i. ${}_{10}p_{500}$

   $${}_{10}p_{500} = e^{-10c}$$

j. Would this be a reasonable model for human mortality? Why or why not?

This would be an unreasonable model. One would expect that as a person ages, the probability of surviving another 10 years would decrease in general. But Part g, h, and i suggest that they are the same.

4. You are given ${}_t q_0 = \dfrac{t^2}{10,000}$ for $0 < t < 100$. Calculate:

a. $F_0(x)$

$$F_0(x) = {}_x q_0 = \frac{x^2}{10000} \quad \text{for } 0 < t < 100$$

b. $S_0(x)$

$$S_0(x) = {}_x p_0 = 1 - {}_x q_0 = 1 - \frac{x^2}{10000} \quad \text{for } 0 < t < 100$$

c. $S_x(t)$

$$S_x(t) = {}_t p_x = \frac{{}_{t+x} p_0}{{}_x p_0} = \frac{1 - \dfrac{(t+x)^2}{10000}}{1 - \dfrac{x^2}{10000}} = \frac{10000 - (t+x)^2}{10000 - x^2} \quad \text{for } 0 < t < 100 - x$$

d. $f_0(x)$

$$f_0(x) = -\frac{d}{dx} S_0(x) = -\frac{d}{dx}\left(1 - \frac{x^2}{10000}\right) = \frac{x}{5000}$$

e. $E[T_0]$

$$E[T_0] = \int_{t=0}^{t=100} {}_t p_0 \, dt = \int_{t=0}^{t=100} 1 - \frac{t^2}{10000} dt = \left[t - \frac{t^3}{30000} \right]_{t=0}^{t=100} = \frac{200}{3} \approx 66.67$$

f. $Var[T_0]$

$$E\left[T_0^{\,2}\right] = 2\int_{t=0}^{t=100} t \cdot {}_t p_0 \, dt = 2\int_{t=0}^{t=100} t \cdot \left(1 - \frac{t^2}{10000}\right) dt$$

$$= 2\int_{t=0}^{t=100} t - \frac{t^3}{10000} dt = 2\left[\frac{t^2}{2} - \frac{t^4}{40000} \right]_{t=0}^{t=100} = 5000$$

$$Var[T_0] = E\left[T_0^{\,2}\right] - \left(E[T_0]\right)^2 = 5000 - \left(\frac{200}{3}\right)^2 = \frac{5000}{9} \approx 555.56$$

g. ${}_{40} q_0$

$${}_{40} q_0 = \frac{40^2}{10000} = 0.16$$

h. ${}_{40} p_0$

$${}_{40} p_0 = 1 - {}_{40} q_0 = 0.84$$

i. $\Pr(40 < T_0 < 60)$

$${}_{60} q_0 - {}_{40} q_0 = \frac{60^2}{10000} - \frac{40^2}{10000} = 0.2$$

j. μ_x

$$\mu_x = \frac{f_0(x)}{S_0(x)} = \frac{\frac{x}{5000}}{1 - \frac{x^2}{10000}} = \frac{2x}{10000 - x^2} \quad \text{for } 0 < t < 100$$

k. μ_{75}

$$\mu_{75} = \frac{(2)(75)}{10000 - 75^2} = \frac{6}{175}$$

l. ${}_t p_x$

$$ {}_t p_x = S_x(t) = \frac{10000 - (t+x)^2}{10000 - x^2} \quad \text{for } 0 < t < 100 - x$$

m. ${}_t q_x$

$$ {}_t q_x = 1 - {}_t p_x = \frac{(t+x)^2 - x^2}{10000 - x^2} = \frac{2tx + t^2}{10000 - x^2} \quad \text{for } 0 < t < 100 - x$$

n. ${}_t q_{75}$

$$ {}_t q_{75} = \frac{150t + t^2}{10000 - 75^2} = \frac{150t + t^2}{4375} \quad \text{for } 0 < t < 25$$

o. ${}_t p_{75}$

$$ {}_t p_{75} = \frac{10000 - (75+t)^2}{10000 - 75^2} = \frac{4375 - 150t - t^2}{4375} \quad \text{for } 0 < t < 25$$

p. $E[T_x]$

$$E[T_x] = \int_{t=0}^{t=100-x} {}_t p_x \, dt = \int_{t=0}^{t=100-x} \frac{10000 - x^2 - 2tx - t^2}{10000 - x^2} dt$$

$$= \frac{1}{10000 - x^2} \cdot \left[\left(10000 - x^2\right) t - t^2 x - \frac{t^3}{3} \right]_{t=0}^{t=100-x}$$

$$= \frac{1}{10000 - x^2} \cdot \left[\left(10000 - x^2\right)(100 - x) - x(100 - x)^2 - \frac{1}{3} \cdot (100 - x)^3 \right]$$

$$= \frac{1}{100 + x} \cdot \left[\left(10000 - x^2\right) - x(100 - x) - \frac{1}{3}(100 - x)^2 \right]$$

$$= \frac{(100 - x)}{3(100 + x)} \cdot \left[3(100 + x) - 3x - (100 - x) \right] = \frac{(100 - x)(200 + x)}{300 + 3x}$$

q. $E[T_{75}]$

$$E\left[T_{75}\right]=\frac{(100-75)\cdot(200+75)}{300+3\times 75}=\frac{275}{21}=13.09524$$

r. $\overset{\circ}{e}_x$

$$\overset{\circ}{e}_x=E\left[T_x\right]=\frac{(100-x)(200+x)}{300+3x}$$

s. $\overset{\circ}{e}_0$

$$\overset{\circ}{e}_0=\frac{100\times 200}{300}=\frac{200}{3}=66.67$$

t. $\overset{\circ}{e}_{75}$

$$\overset{\circ}{e}_{75}=\frac{(100-75)\cdot(200+75)}{300+3\times 75}=\frac{275}{21}=13.09524$$

u. $\overset{\circ}{e}_{75:\overline{10}|}$

$$\overset{\circ}{e}_{75:\overline{10}|}=\int_{t=0}^{t=10}{}_{t}p_{75}dt=\int_{t=0}^{t=10}\frac{10000-75^2-150t-t^2}{10000-75^2}dt$$

$$=\frac{1}{4375}\cdot\left[4375t-75t^2-\frac{t^3}{3}\right]_{t=0}^{t=10}=8.20952$$

v. ${}_{u|t}q_x$

$${}_{u|t}q_x={}_{u}p_x\cdot{}_{t}q_{x+u}$$

$$=\frac{10000-(u+x)^2}{10000-x^2}\cdot\frac{(t+u+x)^2-(u+x)^2}{10000-(u+x)^2}=\frac{(t+u+x)^2-(u+x)^2}{10000-x^2}\quad\text{for } 0<t<100-x-u$$

w. ${}_{10|5}q_{75}$

$${}_{10|5}q_{75}=\frac{(5+10+75)^2-(10+75)^2}{10000-75^2}=0.2$$

5. You are given that $\mu_x = \dfrac{2}{100-x}$ for $0 \le x < 100$. Calculate $F_0(x)$ and ${}_{10}p_{50}$.

$$ {}_x p_0 = e^{-\int \mu_x \, dx} = e^{-\int_{t=0}^{t=x} \frac{2}{100-t} dt} $$

$$ = e^{2\left[\ln(100-t)\right]_{t=0}^{t=x}} = e^{2\left[\ln(100-x)-\ln(100)\right]} $$

$$ = \frac{e^{2\ln(100-x)}}{e^{2\ln(100)}} = \frac{(100-x)^2}{100^2} $$

$$ F_0(x) = 1 - {}_x p_0 = 1 - \frac{(100-x)^2}{100^2} $$

$$ {}_{10}p_{50} = \frac{{}_{60}p_0}{{}_{50}p_0} = \frac{\frac{(100-60)^2}{100^2}}{\frac{(100-50)^2}{100^2}} = \frac{40^2}{50^2} = 0.64 $$

6. Given that $p_x = 0.99$, $p_{x+1} = 0.985$, ${}_3p_{x+1} = 0.95$, and $q_{x+3} = 0.02$. Calculate

   a. p_{x+3}

      $p_{x+3} = 1 - q_{x+3} = 0.98$

   b. ${}_2p_x$

      ${}_2p_x = p_x \cdot p_{x+1} = (0.99)(0.985) = 0.97515$

   c. ${}_2p_{x+1}$

      ${}_3p_{x+1} = {}_2p_{x+1} \cdot p_{x+3}$

      $==>{}_2p_{x+1} = \dfrac{{}_3p_{x+1}}{p_{x+3}} = \dfrac{0.95}{0.98} = 0.96939$

   d. ${}_3p_x$

      ${}_3p_x = p_x \cdot p_{x+1} \cdot p_{x+2} = p_x \cdot {}_2p_{x+1} = (0.99)(0.96939) = 0.95969$

   e. ${}_{1|2}q_x$

      ${}_{1|2}q_x = p_x \cdot {}_2q_{x+1} = p_x \cdot (1 - {}_2p_{x+1}) = 0.99 \times (1 - 0.96939) = 0.030306$

7.

a. $${}_t p_x^{sm} = \exp\left(\int_0^t \mu_{x+s}^{sm} \cdot ds\right) = \exp\left(\int_0^t 2 \cdot \mu_{x+s}^{ns} \cdot ds\right) = \left({}_t p_x^{ns}\right)^2$$

b. Answers shall be presented in EXCEL.

c. Answers shall be presented in EXCEL

8. You are given the following mortality table:

x	q_x for males	q_x for females
90	0.20	0.10
91	0.25	0.15
92	0.30	0.20
93	0.40	0.25
94	0.50	0.30
95	0.60	0.40
96	1.00	1.00

a. Calculate the probability that a male exact age 91 will die at age 93 or 94.

$${}_2p_{91}^M \cdot {}_2q_{93}^M = p_{91}^M \cdot p_{92}^M \cdot \left(1 - p_{93}^M \cdot p_{94}^M\right)$$

$$= (1 - 0.25)\cdot(1 - 0.3)\cdot\left[1 - (1 - 0.4)(1 - 0.5)\right]$$

$$= 0.3675$$

b. Calculate the amount that the curtate life expectancy for a female age 90 exceeds the curtate life expectancy for a male age 90.

$$e_{90}^F = \sum_{t=1}^{t=\infty} {}_tp_{90}^F$$

$$= {}_1p_{90}^F + {}_2p_{90}^F + {}_3p_{90}^F + {}_4p_{90}^F + {}_5p_{90}^F + {}_6p_{90}^F$$

$$= {}_1p_{90}^F$$

$$+ {}_1p_{90}^F \cdot {}_1p_{91}^F$$

$$+ {}_1p_{90}^F \cdot {}_1p_{91}^F \cdot {}_1p_{92}^F$$

$$+ {}_1p_{90}^F \cdot {}_1p_{91}^F \cdot {}_1p_{92}^F \cdot {}_1p_{93}^F$$

$$+ {}_1p_{90}^F \cdot {}_1p_{91}^F \cdot {}_1p_{92}^F \cdot {}_1p_{93}^F \cdot {}_1p_{94}^F$$

$$+ {}_1p_{90}^F \cdot {}_1p_{91}^F \cdot {}_1p_{92}^F \cdot {}_1p_{93}^F \cdot {}_1p_{94}^F \cdot {}_1p_{95}^F$$

$$+ {}_1p_{90}^F \cdot {}_1p_{91}^F \cdot {}_1p_{92}^F \cdot {}_1p_{93}^F \cdot {}_1p_{94}^F \cdot {}_1p_{95}^F \cdot {}_1p_{96}^F$$

$$= {}_1p_{90}^F \cdot \left(1 + \left({}_1p_{91}^F \cdot \left(1 + {}_1p_{92}^F \cdot \left(1 + {}_1p_{93}^F \cdot \left(1 + {}_1p_{94}^F \cdot \left(1 + {}_1p_{95}^F \cdot \left(1 + {}_1p_{96}^F\right)\right)\right)\right)\right)\right)\right)$$

$$= 0.9 \cdot \left(1 + \left(0.85 \cdot \left(1 + 0.8 \cdot \left(1 + 0.75 \cdot \left(1 + 0.7 \cdot \left(1 + 0.6 \cdot (1 + 0)\right)\right)\right)\right)\right)\right)$$

$$= 3.25008$$

$$e_{90}^{M} = {}_1p_{90}^{M} \cdot \left(1 + \left({}_1p_{91}^{M} \cdot \left(1 + {}_1p_{92}^{M} \cdot \left(1 + {}_1p_{93}^{M} \cdot \left(1 + {}_1p_{94}^{M} \cdot \left(1 + {}_1p_{95}^{M} \cdot \left(1 + {}_1p_{96}^{M}\right)\right)\right)\right)\right)\right)\right)$$

$$= 0.8 \cdot \left(1 + \left(0.75 \cdot \left(1 + 0.7 \cdot \left(1 + 0.6 \cdot \left(1 + 0.5 \cdot \left(1 + 0.4 \cdot \left(1 + 0\right)\right)\right)\right)\right)\right)\right)$$

$$= 2.2484$$

$$\Delta e = e_{90}^{F} - e_{90}^{M} = 1.00168$$

c. For females, calculate $e_{91:\overline{3}|}$.

$$e_{91:\overline{3}|}^{F} = \sum_{t=1}^{3} {}_tp_{91} = {}_1p_{91} + {}_2p_{91} + {}_3p_{91}$$

$$= {}_1p_{91}\left(1 + {}_2p_{91}\left(1 + {}_3p_{91}\right)\right)$$

$$= (0.85)\left(1 + (0.8)(1 + 0.75)\right)$$

$$= 2.04$$

9. You are given the following:
   a. $e_{40:\overline{20}|} = 18$
   b. $e_{60} = 25$
   c. ${}_{20}q_{40} = 0.2$
   d. $q_{40} = 0.003$

   Calculate e_{41}

$$e_{40} = e_{40:\overline{20}|} + {}_{20}p_{40} \cdot e_{60}$$
$$= 18 + (1 - 0.2)(25)$$
$$= 38$$

$$e_{40} = e_{40:\overline{1}|} + p_{40} \cdot e_{41}$$
$$==> e_{41} = \frac{e_{40} - e_{40:\overline{1}|}}{p_{40}} = \frac{e_{40} - p_{40}}{p_{40}} = \frac{38 - 0.997}{0.997} = 37.11434$$

10. (SWAQ) You are given that ${}_tp_x = 1 - \frac{t^3}{125}$.

Your boss calculates the complete expectation of life for (x) using the relationship that the complete expectation of life is approximately equal to the curtate expectation of life plus one half of a year.

a. Calculate the complete expectation of life for (x) as computed by your boss.

$$\omega = \text{Maximum Age} = 5$$

$$e_x = \sum_{t=1}^{t=5} {}_tp_x$$

$$= \sum_{t=1}^{t=5} 1 - \frac{t^3}{125}$$

$$= 5 - \frac{1^3 + 2^3 + 3^3 + 4^3 + 5^3}{125}$$

$$= 3.2$$

$$\overset{\circ}{e}{}_x^{Boss} \approx e_x + \frac{1}{2} = 3.7$$

b. Calculate the actual complete expectation of life.

$$\overset{\circ}{e}_x = \int_{t=0}^{t=5} {}_tp_x dt$$

$$= \int_{t=0}^{t=5} 1 - \frac{t^3}{125} dt$$

$$= \left[t - \frac{t^4}{500} \right]_{t=0}^{t=5}$$

$$= 3.75$$

c. Write a short paragraph explaining to your boss why his calculation gets a different answer. Identify the error in his approximation.

The assumption made by the boss: the complete life expectation is equal to the curtate life expectation plus one half. This assumption is a fairly accurate assumption but only when the death is uniform distributed within a complete year. In the real life, this UDD assumption is in generally not true; therefore, the assumption leads to a slight error.