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# A NONCONFORMING MIXED FINITE ELEMENT METHOD FOR MAXWELL'S EQUATIONS

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We present a nonconforming mixed finite element scheme for the approximate solution of the time-harmonic Maxwell's equations in a three-dimensional, bounded domain with absorbing boundary conditions on artificial boundaries. The numerical procedures are employed to solve the direct problem in magnetotellurics consisting in determining a scattered electromagnetic field in a model of the earth having bounded conductivity anomalies of arbitrary shapes. A domain-decomposition iterative algorithm which is naturally parallelizable and is based on a hybridization of the mixed method allows the solution of large three-dimensional models. Convergence of the approximation by the mixed method is proved, as well as the convergence of the iteration.

## 1. Introduction

The magnetotelluric method is used to infer the distribution of the earth's electric conductivity from measurements of natural electric and magnetic fields on the earth's surface. The electromagnetic fields are assumed to obey Maxwell's equations. Their natural sources are electric storms and solar wind fluctuations that generate electromagnetic waves in the ionosphere. These waves arrive at the earth's

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surface as plane incident waves and generate *telluric* currents. Applications of the magnetotelluric method include petroleum exploration and detection of groundwater reservoirs and mineral deposits [2].

The object of this paper is to present a numerical procedure to determine the scattered electromagnetic fields induced inside the earth when a plane electromagnetic wave arrives normally to the earth's surface; the earth is modelled as a horizontally-layered medium containing arbitrarily shaped conductivity anomalies. The numerical scheme is a new nonconforming mixed finite element procedure that can be hybridized in a manner leading to a domain-decomposition iterative technique to solve the algebraic equations associated with the procedure.

Numerical methods to solve the direct problem in magnetotellurics have been proposed previously by several authors. In a classical work by P. E. Wannamaker *et al.* [27], a finite element method was employed to solve the two-dimensional scattering problem formulated as the time-harmonic Maxwell's equations considered as a set of two second-order elliptic equations. Their method requires the calculation of derivatives of the conductivity coefficient, introducing unnecessary numerical complexity. A moving finite element procedure to solve the two-dimensional magnetotelluric problem was presented by B. Travis *et al.* [26]. A finite difference procedure for three-dimensional magnetotellurics was presented by R. L. Mackie *et al.* [14].

Finite difference algorithms to solve Maxwell's equations in the time-domain have been widely used in electrical engineering applications, with the best-known procedure being due to K. Yee [28]. A convergence analysis for Yee's scheme was given by P. Monk [19], who also discussed finite element procedures for Maxwell's equations in two and three dimensions in several papers (see, e.g., [15,16,17,18]). A collection of mixed finite element methods for two-dimensional magnetotellurics has been recently presented in [21,22].

In this work we develop a nonconforming mixed finite element scheme for solving the time-harmonic Maxwell's equations in a bounded domain with absorbing boundary conditions on artificial exterior boundaries. The method employs a nonconforming element discussed in [8] for solving second-order elliptic problems and is hybridized to enable the development of a domain-decomposition iterative method based on Robin transmission conditions; methods of this type were discussed by Lions [12,13] and extended to the complex-valued, noncoercive Helmholtz problem by Després [5] and to a different numerical method for Maxwell's equations by Després, Joly, and Roberts [6]. Improved estimates of the rate of convergence for these methods were established in [7] for mixed methods for second-order elliptic problems and the analysis extended to a nonconforming method in [8]. The hybridization method used in all of these papers is based on ideas of Fraeijs de Veubeke [10,11], as interpreted by Arnold and Brezzi [1]. The domain decomposition algorithm is naturally parallelizable and provides a fast technique for the solution of the time-harmonic Maxwell's equations in large, three-dimensional, inhomogeneous earth conductivity models.

The organization of the paper is as follows. In §2 we describe the physical problem and the differential equations and boundary conditions employed for its mathematical description. Also, some necessary notation is introduced. In §3 the nonconforming mixed finite element spaces used for the spatial discretization are presented and their approximation properties analyzed. A mixed weak formulation of the problem is also presented and analyzed. In §4 we demonstrate the equivalence between the hybridized mixed finite element method and the underlying nonconforming method and derive a set of *a priori* error estimates. Finally, in §5 we present a domain-decomposition iterative procedure for the hybridized nonconforming mixed method and derive its convergence.

## 2. The Differential Model

If $E$ and $H$ denote the electric and magnetic fields for a given angular frequency $\omega$, the time-harmonic Maxwell's equations in a region free of sources are given by

$$(\sigma + i\omega\varepsilon)E - \nabla \times H = 0, \tag{2.1a}$$

$$i\omega\mu H + \nabla \times E = 0, \tag{2.1b}$$

where $\sigma$, $\varepsilon$, and $\mu$ denote the conductivity, electric permittivity, and magnetic permeability, respectively. In magnetotelluric modelling the medium parameters $\sigma$, $\varepsilon$, and $\mu$ are usually assumed as follows, which will also be imposed in what follows:

- $\mu$ is close to $\mu_0 = 4\pi 10^{-7} Henry/m$, the magnetic permeability of a vacuum;
- $\varepsilon$ is close to $10\varepsilon_0 = 10 \cdot \dfrac{1}{36\pi} 10^{-10} Coulomb^2/Newton^2$, $\varepsilon_0$ being the electric permittivity of a vacuum;
- $\sigma$ is bounded below and above by positive constants.

The terms $\sigma E$ and $i\omega\varepsilon E$ in (2.1a) represent conductivity and displacement currents, respectively. In magnetotellurics, a limited range of frequencies and conductivity values are of interest, more precisely, $f = \omega/2\pi \leq 100$ Hz and $\sigma \in [.001, 1]$ (in $(1/ohm \cdot m)$-units), so that

$$\omega\varepsilon << \sigma,$$

and displacement currents can be neglected. Thus, Maxwell's equations reduce to

$$\sigma E - \nabla \times H = 0, \tag{2.2a}$$

$$i\omega\mu H + \nabla \times E = 0. \tag{2.2b}$$

Our differential model is formulated in terms of scattered fields (see also [4]). Consider the primary model, without a scatterer, identified with $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ where the medium parameters $\sigma$ and $\mu$ are assumed to have known values $\sigma_p$ and $\mu_p$, respectively. (In magnetotellurics, $\sigma_p$ and $\mu_p$ are usually assumed to represent a layered medium.) Suppose that a bounded scatterer $\Omega_s$ is embedded in $\mathbb{R}^3_+$; the primary medium parameters $\sigma_p$ and $\mu_p$ in $\Omega_s$ are then changed into the parameters $\sigma = \sigma_p + \sigma_s$ and $\mu = \mu_p + \mu_s$ with $\mathrm{supp}(\sigma_s) \cup \mathrm{supp}(\mu_s) \subset \Omega_s$. Let the primary electromagnetic fields $E_p$ and $H_p$ be physically meaningful solutions of Maxwell's equations (2.2) in $\mathbb{R}^3_+$ for the primary model. Then, let $E_t = E_p + E_s$ and $H_t = H_p + H_s$ denote the total electromagnetic fields in $\mathbb{R}^3_+$ with $\sigma$ and $\mu$ induced by a plane, monochromatic electromagnetic wave of frequency $\omega$ incident upon the boundary $z = 0$ of $\mathbb{R}^3_+$. Finally, let $E_s$ and $H_s$ be the scattered electromagnetic fields due to the presence of the anomalies of $\Omega_s$; they satisfy the equations

$$\begin{aligned} \sigma E_s - \nabla \times H_s &= -\sigma_s E_p \equiv F \quad \text{in } \mathbb{R}^3_+, \\ i\omega\mu H_s + \nabla \times E_s &= -i\omega\mu_s H_p \equiv G \quad \text{in } \mathbb{R}^3_+. \end{aligned}$$

Truncate the problem to a compact domain, so that a practical computational procedure can be defined. Let $\Omega \subset \mathbb{R}^3_+$ be a cube containing $\Omega_s$ and big enough so that $\Gamma \equiv \partial\Omega$ is far away from $\Omega_s$. Without loss of generality, the problem can be scaled so that $\Omega$ is the unit cube whose bottom face is included in the boundary $z = 0$ of $\mathbb{R}^3_+$. Now, consider the scattering problem to find $(E, H) \equiv (E_s, H_s)$:

$$\sigma E - \nabla \times H = F \quad \text{in } \Omega, \tag{2.3a}$$

$$i\omega\mu H + \nabla \times E = G \quad \text{in } \Omega, \tag{2.3b}$$

for given $F$ and $G$. To minimize the effect of reflections from the artificial boundary $\Gamma$, impose the absorbing boundary condition

$$(1 - i)P_\tau aE + \nu \times H = 0 \quad \text{on } \Gamma, \qquad a = [\sigma/(2\omega\mu)]^{1/2}, \tag{2.4}$$

where $\nu$ denotes the unit outer normal to $\Gamma$ and $P_\tau\varphi = \varphi - \nu(\nu \cdot \varphi) = -\nu \times (\nu \times \varphi)$ is the projection of the trace of $\varphi$ on $\Gamma$.

Assume that

$$0 < \sigma_{\min} \le \sigma \le \sigma_{\max}, \qquad 0 < \mu_{\min} \le \mu \le \mu_{\max}.$$

Also, assume that $a$ is a real-valued, Lipschitz-continuous function on $\Gamma$ such that $0 < a_{\min} \le a(x)$ for $x \in \Gamma$.

A proof of the following existence and uniqueness results for (2.3)–(2.4) is given in [23].

**Theorem 2.1** *Let $F, G \in [L^2(\Omega)]^3$ and $\omega \ne 0$. Then, there exists a unique electromagnetic field $(E, H) \in [H(curl; \Omega)]^2$ satisfying (2.3)-(2.4). If, in addition, $F$ and $G$ belong to $H(div; \Omega)$ and $\sigma$ and $\mu$ are Lipschitz-continuous on $\overline{\Omega}$, then $E$ and $H$ belong to $[H^{1/2}(\Omega)]^3$; more precisely, $\{E, H\} \in [H(curl; \Omega) \cap H(div; \Omega)]$ with boundary values in $[L^2(\Gamma)]^6$.*

Let $(H^s(\Omega), \|\cdot\|_s)$ and $(H^s(\Gamma), |\cdot|_s)$ indicate standard, complex Sobolev spaces for any real number $s$, and set $H^0(\Omega) = L^2(\Omega)$; $\|\cdot\|_{0,\Omega} = \|\cdot\|$ denotes the usual $L^2$-norm with the associated inner product

$$(\varphi, \psi) = \int_\Omega \varphi\overline{\psi}\, dx\, dy\, dz.$$

Also, for a face $f$ of $\Omega$, let

$$\langle\varphi, \psi\rangle_f = \int_f \varphi\overline{\psi}\, df$$

denote the inner product on $L^2(f)$, with associated norm $|\cdot|_{0,f}$. Let the Hilbert space

$$H(curl; \Omega) = \left\{\varphi \in [L^2(\Omega)]^3 : \nabla \times \varphi \in [L^2(\Omega)]^3\right\}$$

be equipped with the natural norm and inner product

$$\|\varphi\|_{H(curl;\Omega)} = (\|\varphi\|_0^2 + \|\nabla \times \varphi\|_0^2)^{\frac{1}{2}}, \qquad (\varphi, \psi)_{H(curl;\Omega)} = (\varphi, \psi) + (\nabla \times \varphi, \nabla \times \psi).$$

Denote by $Lip(\Gamma)$ the space of all Lipschitz-continuous functions on $\Gamma$ and by $Lip(\Gamma)'$ the dual space of $Lip(\Gamma)$. It is shown in [24] that $(\nu \times \varphi) \cdot P_\tau\psi \in Lip(\Gamma)'$ for all $\varphi, \psi \in H(curl; \Omega)$.

The following generalized Green's formula on $H(curl;\Omega)$ [24,25] will be useful:

$$(\nabla\times\varphi,\psi)-(\varphi,\nabla\times\psi)=\langle\nu\times\varphi,\psi\rangle_\Gamma=\langle\nu\times\varphi,P_\tau\psi\rangle_\Gamma,\quad \forall\varphi,\psi\in H(curl;\Omega), \tag{2.5}$$

where the boundary integral term $\langle\nu\times\varphi,P_\tau\psi\rangle_\Gamma$ is understood as $\langle(\nu\times\varphi)\cdot\overline{P_\tau\psi},1\rangle$, the duality pairing between $\nu\times\varphi\cdot\overline{P_\tau\psi}\in Lip(\Gamma)'$ and $1\in Lip(\Gamma)$. Note that $\nu\times\varphi$ and $P_\tau\psi$ belong only to $[H^{-1/2}(\Gamma)]^3$ for $\varphi,\psi\in H(curl;\Omega)$.

Introduce $H^*(curl;\Omega)=\{\varphi\in H(curl;\Omega);P_\tau\alpha\varphi=\nu\times\chi$ on $\partial\Omega$ for some $\chi\in H(curl;\Omega)\}=\{\varphi\in H(curl;\Omega);\nu\times\varphi\in[L^2(\partial\Omega)]^3\}$. Test (2.3a) and (2.3b) against $\varphi\in H^*(curl;\Omega)$ and $\psi\in[L^2(\Omega)]^3$ and apply (2.5) to obtain the mixed, weak problem of finding $(E,H)\in H^*(curl;\Omega)\times[L^2(\Omega)]^3$ such that

$$(\sigma E,\varphi)-(H,\nabla\times\varphi)+(1-i)\langle P_\tau aE,P_\tau\varphi\rangle_\Gamma=(F,\varphi),\quad \varphi\in H^*(curl;\Omega), \tag{2.6a}$$
$$i\omega(\mu H,\psi)+(\nabla\times E,\psi)=(G,\psi),\qquad \psi\in[L^2(\Omega)]^3. \tag{2.6b}$$

We shall approximate (2.6) by a nonconforming method, which can be hybridized and then solved by domain decomposition iterative methods.

## 3. A Nonconforming Mixed Finite Element Procedure

For $0<h<1$, let $\mathcal{T}^h$ be a quasiregular partition of $\Omega$ into three-dimensional rectangles $\Omega_j$, $j=1,\cdots,J$, with diameters bounded by $h$:

$$\overline{\Omega}=\cup_j\overline{\Omega}_j,\quad \Omega_j\cap\Omega_k=\phi,\quad j\neq k.$$

Let $\widehat{K}$ be the cube $[-1,1]^3$ and let $\widehat{Q}=\widehat{Q}_x\times\widehat{Q}_y\times\widehat{Q}_z$, where (see [8])

$$\widehat{Q}_x=\text{Span}\left\{1,y,z,\left(y^2-\frac{5}{3}y^4\right)-\left(z^2-\frac{5}{3}z^4\right)\right\},$$
$$\widehat{Q}_y=\text{Span}\left\{1,z,x,\left(z^2-\frac{5}{3}z^4\right)-\left(x^2-\frac{5}{3}x^4\right)\right\},$$
$$\widehat{Q}_z=\text{Span}\left\{1,x,y,\left(x^2-\frac{5}{3}x^4\right)-\left(y^2-\frac{5}{3}y^4\right)\right\}.$$

Let $\xi_i$, $i=1,\ldots,6$, be the centroid of the $i^{th}$ face of $\widehat{K}$. For $\varphi\in\widehat{Q}(\widehat{K})$, consider the following local degrees of freedom:

$$\sum(\varphi)=\{(P_\tau\varphi)(\xi_i),\ i=1,\cdots,6\}. \tag{3.1}$$

Define a local interpolant $\widehat{\pi}:[H^2(\widehat{K})]^3\to\widehat{Q}(\widehat{K})$ as follows:

$$P_\tau(\widehat{\pi}\varphi-\varphi)(\xi_i)\quad=\quad 0,\qquad i=1,\cdots,6. \tag{3.2}$$

Note that (3.2) provides the twelve degrees of freedom needed to determine an element in $\widehat{Q}(\widehat{K})$. Next, let $\widehat{S}=\widehat{S}_x\times\widehat{S}_y\times\widehat{S}_z$, where

$$\widehat{S}_x=\text{Span}\left\{1,y-\frac{10}{3}y^3,z-\frac{10}{3}z^3\right\},$$
$$\widehat{S}_y=\text{Span}\left\{1,z-\frac{10}{3}z^3,x-\frac{10}{3}x^3\right\},$$
$$\widehat{S}_z=\text{Span}\left\{1,x-\frac{10}{3}x^3,y-\frac{10}{3}y^3\right\},$$

and define a local interpolant $\widehat{P} : [L^2(\widehat{K})]^3 \to \widehat{S}(\widehat{K})$ as follows. For $\psi = (\psi_x, \psi_y, \psi_z)$,

$$\int_{\widehat{K}} (\widehat{P}\psi_\ell - \psi_\ell)dx\,dy\,dz = 0, \qquad \int_{\widehat{K}} curl(\widehat{P}\psi_\ell - \psi_\ell)dx\,dy\,dz = 0, \quad \ell = x, y, z, \quad (3.3)$$

where, in (3.3), the two-dimensional *curl* is defined as usual:

$$curl\psi_x = \left(\frac{\partial \psi_x}{\partial z}, -\frac{\partial \psi_x}{\partial y}\right), \quad curl\psi_y = \left(\frac{\partial \psi_y}{\partial x}, -\frac{\partial \psi_y}{\partial z}\right), \quad curl\psi_z = \left(\frac{\partial \psi_z}{\partial y}, -\frac{\partial \psi_z}{\partial x}\right).$$

Note that (3.3) provides the nine degrees of freedom needed to determine an element in $\widehat{S}(\widehat{K})$ and that

$$\nabla \times \widehat{Q} = \widehat{S}.$$

The following proposition states an immediate but fundamental property of $\widehat{Q}$ and $\widehat{S}$ that is important in obtaining effective nonconforming methods [8].

**Proposition 3.1** *If $P_\tau \widehat{Q}$ or $P_\tau \widehat{S}$ vanishes at the center of a face of $\widehat{K}$, it is orthogonal to constants on that face.*

The following lemma is trivial.

**Lemma 3.1** *The degrees of freedom* (3.2) *and* (3.3) *determine, respectively, $\varphi \in \widehat{Q}(\widehat{K})$ and $\psi \in \widehat{S}(\widehat{K})$ uniquely.*

Define $Q(\Omega_j)$ and $S(\Omega_j)$ by scaling and translating from $\widehat{Q}$ and $\widehat{S}$.
Let

$$\Gamma_j = \partial\Omega_j \cap \Gamma, \qquad \Gamma_{jk} = \partial\Omega_j \cap \partial\Omega_k = \Gamma_{kj},$$

and

$$\widetilde{\Lambda}^h = \left\{ \widetilde{\lambda}^h : \widetilde{\lambda}^h|_{\Gamma_{jk}} = \widetilde{\lambda}_{jk} \in P_0 \times P_0 \text{ for each face } \Gamma_{jk} \text{ of } \Omega_j;\ \widetilde{\lambda}_{jk} + \widetilde{\lambda}_{kj} = 0 \right\}.$$

Denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\Gamma_{jk}}$ the approximation to $\langle \cdot, \cdot \rangle_{\Gamma_{jk}}$ obtained by using the mid-point rule on $\Gamma_{jk}$; i.e., if $\xi_{jk}$ denotes the centroid of $\Gamma_{jk}$, then

$$\langle\!\langle u, v \rangle\!\rangle_{\Gamma_{jk}} = |\Gamma_{jk}|(u\overline{v})(\xi_{jk}), \qquad |\Gamma_{jk}| \text{ being the measure of } \Gamma_{jk}.$$

Define the nonconforming mixed finite element space $V^h \times W^h$ as follows:

$$V^h = \left\{ \varphi \in [L^2(\Omega)]^3 : \varphi|_{\Omega_j} \in Q(\Omega_j) \text{ and } \sum_{jk} \langle\!\langle \theta, P_\tau \varphi \rangle\!\rangle_{\Gamma_{jk}} = 0, \quad \forall \theta \in \widetilde{\Lambda}^h \right\},$$

$$W^h = \left\{ \psi \in [L^2(\Omega)]^3 : \psi|_{\Omega_j} \in S(\Omega_j) \right\},$$

and set

$$V_j^h = V^h|_{\Omega_j} \quad \text{and} \quad W_j^h = W^h|_{\Omega_j}.$$

Let $\pi_h : [H^2(\Omega)]^3 \to V^h$ be the interpolation operator such that $\pi_h|_{\Omega_j}$ is defined by the degrees of freedom (3.2), and let $\mathbf{P}_h : [L^2(\Omega)]^3 \to W^h$ be the $L^2$-projection operator defined by

$$(\psi - \mathbf{P}_h\psi, \chi) = 0, \qquad \forall \chi \in W^h. \tag{3.4}$$

Since $\nabla \times V_j^h = W_j^h$,

$$\sum_j (\psi - \mathbf{P}_h\psi, \nabla \times \varphi)_j = 0, \qquad \forall \varphi \in V^h. \tag{3.5}$$

Let broken norms and seminorms be defined by

$$\|u\|_{m,h}^2 = \sum_j \|u\|_{m,\Omega_j}^2, \quad |u|_{m,h}^2 = \sum_j |u|_{m,\Omega_j}^2, \quad |u|_{m,h,\Gamma}^2 = \sum_j |u|_{m,\Gamma_j}^2.$$

**Lemma 3.2** [3,20] *Assume that the family of partitions $\mathcal{T}^h$, $0 < h < 1$, is quasiregular. Then, there exists a constant $C$, independent of $h$, such that*

$$\begin{aligned} \|\varphi - \pi_h\varphi\|_0 &\leq Ch\|\varphi\|_2, & \text{(3.6a)} \\ \|\nabla \times (\varphi - \pi_h\varphi)\|_{0,h} &\leq Ch\|\nabla \times \varphi\|_1, & \text{(3.6b)} \\ \|\psi - \mathbf{P}_h\psi\|_0 &\leq Ch\|\psi\|_1. & \text{(3.6c)} \end{aligned}$$

Then, our nonconforming mixed finite element procedure is to find $(E^h, H^h) \in V^h \times W^h$ such that

$$(\sigma E^h, \varphi) - \sum_j (H^h, \nabla \times \varphi)_j + (1-i)\langle\!\langle P_\tau a E^h, P_\tau \varphi \rangle\!\rangle_\Gamma = (F, \varphi), \quad \varphi \in V^h, \tag{3.7a}$$

$$i\omega(\mu H^h, \psi) + \sum_j (\nabla \times E^h, \psi)_j = (G, \psi), \qquad \psi \in W^h. \tag{3.7b}$$

Below, we analyze (3.7), along with a hybridization of it and a domain decomposition iterative procedure. We assume that the solutions $(E, H)$ of (2.6) belong to $[H^2(\Omega)]^3 \times [H^1(\Omega)]^3$ in all that follows.

## 4. Convergence of the Nonconforming Mixed Finite Element Procedure

Denote the boundary truncation error by $G(f,g) = \langle f, g\rangle_\Gamma - \langle\!\langle f, g \rangle\!\rangle_\Gamma$. Then, integration by parts element-by-element shows that

$$\begin{aligned} &(\sigma E, \varphi) - \sum_j (H, \nabla \times \varphi)_j + (1-i)\langle\!\langle P_\tau a E, P_\tau \varphi \rangle\!\rangle_\Gamma & \text{(4.1)} \\ &\quad = (F, \varphi) + \sum_j \langle \nu_j \times H_j, P_\tau \varphi \rangle_{\partial\Omega_j \setminus \Gamma_j} - (1-i)G(P_\tau a E, P_\tau \varphi), \quad \varphi \in V^h, \end{aligned}$$

$$i\omega(\mu H, \psi) + \sum_j (\nabla \times E, \psi)_j = (G, \psi), \quad \psi \in W^h. \tag{4.2}$$

Set

$$e_h = \pi_h E - E^h, \qquad \eta_h = \mathbf{P}_h H - H^h.$$

Since $\nabla \times \varphi \in W^h$ for any $\varphi \in V^h$, the errors $e_h$ and $\eta_h$ satisfy

$$\begin{aligned}
&(\sigma e_h, \varphi) - \sum_j (\eta_h, \nabla \times \varphi)_j + (1-i)\langle\!\langle P_\tau a e_h, P_\tau \varphi \rangle\!\rangle_\Gamma \\
&\quad = (\sigma(\pi_h E - E), \varphi) - \sum_j (\mathbf{P}_h H - H, \nabla \times \varphi)_j + \sum_j \langle \nu_j \times H_j, P_\tau \varphi \rangle_{\partial\Omega_j \setminus \Gamma_j} \\
&\qquad -(1-i)G(P_\tau a E, P_\tau \varphi) + (1-i)\langle\!\langle P_\tau a(\pi_h E - E), P_\tau \varphi \rangle\!\rangle_\Gamma \\
&\quad = (\sigma(\pi_h E - E), \varphi) + \sum_j \langle \nu_j \times H_j, P_\tau \varphi \rangle_{\partial\Omega_j \setminus \Gamma_j} - (1-i)G(P_\tau a E, P_\tau \varphi) \\
&\qquad +(1-i)\langle\!\langle P_\tau a(\pi_h E - E), P_\tau \varphi \rangle\!\rangle_\Gamma, \quad \varphi \in V^h,
\end{aligned} \tag{4.3a}$$

$$\begin{aligned}
&i\omega(\mu\eta_h, \psi) + \sum_j (\nabla \times e_h, \psi)_j \\
&\quad = i\omega\left(\mu(\mathbf{P}_h H - H), \psi\right) + \sum_j \left(\nabla \times (\pi_h E - E), \psi\right)_j, \quad \psi \in W^h.
\end{aligned} \tag{4.3b}$$

Take the test function $\psi = \nabla \times e_h$ in (4.3b):

$$\begin{aligned}
\|\nabla \times e_h\|_{0,h}^2 &= -i\omega \sum_j (\mu\eta_h, \nabla \times e_h)_j + i\omega \sum_j \left(\mu(\mathbf{P}_h H - H), \nabla \times e_h\right)_j \\
&\quad + \sum_j \left(\nabla \times (\pi_h E - E), \nabla \times e_h\right)_j \\
&\le C\big(\|\eta_h\| + \|H\|_1 h + \|\nabla \times E\|_1 h\big)\|\nabla \times e_h\|_{0,h},
\end{aligned}$$

so that

$$\|\nabla \times e_h\|_{0,h} \le C\big(\|\eta_h\| + \|H\|_1 h + \|\nabla \times E\|_1 h\big). \tag{4.4}$$

Next, choose $\varphi = e_h$ in (4.3a) and $\psi = \eta_h$ in (4.3b):

$$\begin{aligned}
&(\sigma e_h, e_h) - \sum_j (\eta_h, \nabla \times e_h)_j + (1-i)\langle\!\langle P_\tau a e_h, P_\tau e_h \rangle\!\rangle_\Gamma \\
&\qquad = (\sigma(\pi_h E - E), e_h) + (1-i)\langle\!\langle a P_\tau(\pi_h E - E), P_\tau e_h \rangle\!\rangle_\Gamma \\
&\qquad\quad + \sum_j \langle \nu_j \times H, P_\tau e_h \rangle_{\partial\Omega_j \setminus \Gamma_j} - (1-i)G(P_\tau a E, P_\tau e_h),
\end{aligned} \tag{4.5a}$$

$$\begin{aligned}
i\omega(\mu\eta_h, \eta_h) + \sum_j (\nabla \times e_h, \eta_h)_j &= i\omega\left(\mu(\mathbf{P}_h H - H), \eta_h\right) \\
&\quad + \sum_j \left(\nabla \times (\pi_h E - E), \eta_h\right)_j.
\end{aligned} \tag{4.5b}$$

Adding (4.5a) and the conjugate of (4.5b) give

$$
\begin{aligned}
(\sigma e_h, e_h) - i\omega(\mu\eta_h, \eta_h) + (1-i)\langle\langle aP_\tau e_h, P_\tau e_h\rangle\rangle_\Gamma \qquad (4.6)\\
= (\sigma(\pi_h E - E), e_h) + (1-i)\langle\langle aP_\tau(\pi_h E - E), P_\tau e_h\rangle\rangle_\Gamma \\
+ \sum_j \langle \nu_j \times H, P_\tau e_h\rangle_{\partial\Omega_j\setminus\Gamma_j} - (1-i)G(P_\tau aE, P_\tau e_h) \\
-i\omega\left(\eta_h, \mu(\mathbf{P}_h H - H)\right) + \sum_j \left(\eta_h, \nabla\times(\pi_h E - E)\right)_j .
\end{aligned}
$$

Let us consider the terms on the right-hand side of (4.6) individually; it is convenient to discuss the third term first. Note that $\nu_j \times H_j + \nu_k \times H_k = 0$ on $\Gamma_{jk}$, so that by Proposition 3.1,

$$
\begin{aligned}
&\langle \nu_j \times H_j, (P_\tau e_h)_j\rangle_{\Gamma_{jk}} + \langle \nu_k \times H_k, (P_\tau e_h)_k\rangle_{\Gamma_{kj}} \\
&\qquad = \langle \nu_j \times H_j, (P_\tau e_h)_j - (P_\tau e_h)_k\rangle_{\Gamma_{jk}} \\
&\qquad = \langle \nu_j \times H_j - \lambda_{jk}, (P_\tau e_h)_j - (P_\tau e_h)_k\rangle_{\Gamma_{jk}} ,
\end{aligned}
$$

where $\lambda_{jk}$ is the restriction to $\Gamma_{jk}$ of an element $\widetilde{\lambda} \in \widetilde{\Lambda}^h$. Hence, if $\lambda_{jk}$ is the $L^2$-projection of $\nu_j \times H_j$ onto $(P_0 \times P_0)(\Gamma_{jk})$, then again by Proposition 3.1,

$$
\begin{aligned}
\left|\sum_j \langle \nu_j \times H_j, P_\tau e_h\rangle_{\partial\Omega_j\setminus\Gamma_j}\right| &= \left|\sum_{j,k} \langle \nu_j \times H_j - \lambda_{jk}, P_\tau e_h\rangle_{\Gamma_{jk}}\right| \\
&\le \sum_{j,k} |\nu_j \times H_j - \lambda_{jk}|_{1/2,\Gamma_{jk}} \, |P_\tau e_h|_{-1/2,\Gamma_{jk}} . \qquad (4.7)
\end{aligned}
$$

By the standard interpolation inequality, for $0 < \delta < 1/2$ and $\theta = 1/[2(1-\delta)]$, we have

$$
\begin{aligned}
\min_{\lambda_{jk}} \sum_k |\nu_j \times H_j - \lambda_{jk}|_{1/2,\Gamma_{jk}} &\le \sum_k |\nu_j \times H_j - \lambda_{jk}|_{1,\Gamma_{jk}}^{1-\theta} |\nu_j \times H_j - \lambda_{jk}|_{\delta,\Gamma_{jk}}^{\theta} \\
&\le C \sum_k |\nu_j \times H_j|_{1,\Gamma_{jk}}^{1-\theta} |\nu_j \times H_j|_{1,\Gamma_{jk}}^{\theta} h^{(1-\delta)\theta} \\
&\le C h^{1/2} \|H_j\|_{3/2,\Omega_j} . \qquad (4.8)
\end{aligned}
$$

Also notice that

$$
|P_\tau e_h|_{-1/2,\Gamma_{jk}} \le C\left[\|e_h\|_{0,\Omega_j} + \|\nabla\times e_h\|_{0,\Omega_j}\right]. \qquad (4.9)
$$

Thus, combining (4.4), (4.7), (4.8), and (4.9), we have

$$
\begin{aligned}
\left|\sum_j \langle \nu_j \times H_j, P_\tau e_h\rangle_{\partial\Omega_j\setminus\Gamma_j}\right| &\le C h^{1/2}\|H\|_{3/2}\left[\|e_h\|_0 + \|\nabla\times e_h\|_0\right] \qquad (4.10)\\
&\le C h^{1/2}\|H\|_{3/2}\left(\|e_h\|_0 + \|\eta_h\|_0 + h\|H\|_1 + h\|\nabla\times\, E\|_1\right) \\
&\le \epsilon\left(\|e_h\|_0^2 + \|\eta_h\|_0^2\right) + Ch\left(\|H\|_{3/2}^2 + h^{1/2}\|\nabla\times\, E\|_1^2\right).
\end{aligned}
$$

Next,

$$
\begin{aligned}
&|(\sigma(\pi_h E - E), e_h)| + |\omega\,(\eta_h, \mu(\mathbf{P}_h H - H))| + \left|\sum_j \left(\eta_h, \nabla\times(\pi_h E - E)\right)_j\right| \\
&\qquad \le C\left(\|\pi_h E - E\|_0^2 + \|P_h H - H\|_0^2 + \|\nabla\times\,(\pi_h E - E)\|_{0,h}^2\right) \\
&\qquad\quad + \epsilon\left(\|e_h\|_0^2 + \|\eta_h\|_0^2\right) \\
&\qquad \le \epsilon\left(\|e_h\|_0^2 + \|\eta_h\|_0^2\right) + Ch^2\left(\|E\|_2^2 + \|H\|_1^2\right).
\end{aligned} \tag{4.11}
$$

By (3.2), the second term in the right side of (4.6) satisfies

$$
\langle\!\langle aP_\tau(\pi_h E - E), P_\tau e_h\rangle\!\rangle_\Gamma = \sum_j \int_{\Gamma_j} aP_\tau(\pi_h E - E)(\xi_j)\cdot\overline{P_\tau e_h(\xi_j)}d\sigma = 0. \tag{4.12}
$$

For the fourth term in the right side of (4.6), it follows from Proposition 3.1 that

$$
\int_{\Gamma_j}\left(\overline{P_\tau e_h(s)} - \overline{P_\tau e_h(\xi_j)}\right)\,ds = 0;
$$

therefore,

$$
\begin{aligned}
&|G(P_\tau aE, P_\tau e_h)| \\
&\quad \le \left|\sum_j \int_{\Gamma_j}\left(a_j(s)P_\tau E(s)\cdot\overline{P_\tau e_h(s)} - a_j(\xi_j)P_\tau E(s)\cdot\overline{P_\tau e_h(s)}\right)\,ds\right| \\
&\qquad + \left|\sum_j \int_{\Gamma_j}\left(a_j(\xi_j)P_\tau E(s)\cdot\overline{P_\tau e_h(s)} - a_j(\xi_j)P_\tau E(\xi_j)\cdot\overline{P_\tau e_h(s)}\right)\,ds\right| \\
&\qquad + \left|\sum_j \int_{\Gamma_j}\left(a_j(\xi_j)P_\tau E(\xi_j)\cdot\overline{P_\tau e_h(s)} - a_j(\xi_j)P_\tau E(\xi_j)\cdot\overline{P_\tau e_h(\xi_j)}\right)\,ds\right| \\
&\quad \le Ch|a|_{1,\infty,\Gamma}|P_\tau E|_{0,\Gamma}\,|P_\tau e_h|_{0,\Gamma} + Ch|a|_{0,\infty,\Gamma}|P_\tau E|_{1,\Gamma}\,|P_\tau e_h|_{0,\Gamma} \\
&\quad \le Ch|E|_{1,\Gamma}\,|P_\tau e_h|_{0,\Gamma}.
\end{aligned} \tag{4.13}
$$

By quasiregularity,

$$
|P_\tau e_h|_{0,\Gamma} \le Ch^{-1/2}\left(\|e_h\|_0 + \|\nabla\times\,e_h\|_{0,h}\right). \tag{4.14}
$$

Combining (4.13), (4.14), and (4.4) gives

$$
\begin{aligned}
|G(P_\tau aE, P_\tau e_h)| &\le Ch|E|_{1,\Gamma}\,|P_\tau e_h|_{0,\Gamma} \\
&\le \epsilon\left(\|e_h\|_0^2 + \|\nabla\times\,e_h\|_0^2\right) + Ch\|E\|_2^2 \\
&\le \epsilon\left(\|e_h\|_0^2 + \|\eta_h\|_0^2 + h^2\|H\|_1^2 + h^2\|\nabla\times\,E\|_1^2\right) + Ch\|E\|_2^2 \\
&\le \epsilon\left(\|e_h\|_0^2 + \|\eta_h\|_0^2\right) + Ch\left(\|E\|_2^2 + h\|H\|_1^2\right).
\end{aligned} \tag{4.15}
$$

Taking the real and imaginary parts in (4.6) and applying the bounds in (4.10), (4.11), (4.12), and (4.15) with an appropriately chosen $\epsilon > 0$ leads to the following estimate:

$$\|e_h\|_0 + \|\eta_h\|_0 + \langle\!\langle P_\tau e_h, P_\tau e_h \rangle\!\rangle^{1/2} \le Ch^{1/2}\left(\|E\|_2 + \|H\|_{3/2}\right). \tag{4.16}$$

Next, by (4.4) and (4.16),

$$\|\nabla \times e_h\|_{0,h} \le Ch^{1/2}\left(\|E\|_2 + \|H\|_{3/2}\right). \tag{4.17}$$

Finally, a combination of the triangle inequality, the approximation properties given in Lemma 3.2, and (4.16)-(4.17) implies the following *a priori* error estimate:

**Theorem 4.1** *Let* $(E, H)$ *and* $(E^h, H^h)$, $0 < h < 1$, *be solutions to* (2.6) *and* (3.7)*, respectively. Then,*

$$\|E - E^h\|_0 + \|H - H^h\|_0 + \|\nabla \times (E - E^h)\|_{0,h} \le Ch^{1/2}\left(\|E\|_2 + \|H\|_{3/2}\right). \tag{4.18}$$

**Remark 4.1** *Theorem 4.1 requires the quasiregularity assumption. However, if an exact quadrature is employed in the computation of the boundary integral term in* (3.7a)*, this assumption is not needed; in fact, no terms related with boundary quadrature errors will appear in the error analysis.*

## 5. The Hybridized Nonconforming Procedure

The hybridization of (3.7) will be performed by associating a space $\widetilde{\Lambda}^h$ of Lagrange multipliers $\widetilde{\lambda}^h$ identified with the value of $\nu \times H$ at the centroid of each face of the elements $\Omega_j$. The nonconforming space $V^h$ will be localized by removing the continuity constraints at the centroids of the interfaces between adjacent elements:

$$\mathcal{NC}^h_{-1} = \{\varphi \in (L^2(\Omega))^3 : \varphi|_{\Omega_j} \in Q(\Omega_j)\}.$$

The hybrid method corresponding to (3.5) consists in finding $(\widetilde{E}^h, \widetilde{H}^h, \widetilde{\lambda}^h) \in \mathcal{NC}^h_{-1} \times W^h \times \widetilde{\Lambda}^h$ such that

$$\begin{aligned}
(\sigma \widetilde{E}^h, \varphi) - \sum_j \left( (\widetilde{H}^h, \nabla \times \varphi)_j + \sum_k \langle\!\langle \widetilde{\lambda}^h, P_\tau \varphi \rangle\!\rangle_{\Gamma_{jk}} \right)& \\
+ (1-i)\langle\!\langle P_\tau a \widetilde{E}^h, P_\tau \varphi \rangle\!\rangle_\Gamma = (F, \varphi),& \qquad \varphi \in \mathcal{NC}^h_{-1}, \qquad (5.1a)\\
i\omega(\mu \widetilde{H}^h, \psi) + \sum_j (\nabla \times \widetilde{E}^h, \psi)_j = (G, \psi),& \qquad \psi \in W^h, \qquad (5.1b)\\
\sum_{jk} \langle\!\langle \theta, P_\tau \widetilde{E}^h \rangle\!\rangle_{\Gamma_{jk}} = 0,& \qquad \theta \in \widetilde{\Lambda}^h. \qquad (5.1c)
\end{aligned}$$

The following lemma is trivial.

**Lemma 5.1** *If $\widetilde{E}^h \in \mathcal{N}C^h_{-1}$, then $\widetilde{E}^h \in V^h$ if and only if*

$$\sum_{jk} \langle\!\langle \theta, P_\tau \widetilde{E}^h \rangle\!\rangle_{\Gamma_{jk}} = 0, \qquad \theta \in \widetilde{\Lambda}^h. \tag{5.2}$$

To show existence and uniqueness of the solution of (5.1), it suffices to show uniqueness. Thus, set $F = G = 0$ in (5.1) and choose $\varphi = \widetilde{E}^h$ in (5.1a), $\psi = \widetilde{H}$ in the conjugate of (5.1b), and $\theta = \widetilde{\lambda}^h$ in (5.1c). Then,

$$(\sigma \widetilde{E}^h, \widetilde{E}^h) - \sum_j (\widetilde{H}^h, \nabla \times \widetilde{E}^h)_j + (1-i)\langle\!\langle P_\tau a \widetilde{E}^h, P_\tau \widetilde{E}^h \rangle\!\rangle_\Gamma = 0, \tag{5.3a}$$

$$-i\omega(\mu \widetilde{H}^h, \widetilde{H}^h) + \sum_j (\widetilde{H}^h, \nabla \times \widetilde{E}^h)_j = 0. \tag{5.3b}$$

Adding (5.3a) and (5.3b) gives

$$(\sigma \widetilde{E}^h, \widetilde{E}^h) - i\omega(\mu \widetilde{H}^h, \widetilde{H}^h) + (1-i)\langle\!\langle P_\tau a \widetilde{E}^h, P_\tau \widetilde{E}^h \rangle\!\rangle_\Gamma = 0. \tag{5.4}$$

Taking the real part in (5.4) implies that $\widetilde{E}^h \equiv 0$, which when substituted into the imaginary part of (5.4) shows that $\widetilde{H}^h \equiv 0$. Then, (5.1a) reduces to

$$\sum_{jk} \langle\!\langle \widetilde{\lambda}^h, P_\tau \varphi \rangle\!\rangle_{\Gamma_{jk}} = 0, \quad \varphi \in \mathcal{N}C^h_{-1}. \tag{5.5}$$

Now, take an element $\widetilde{\Omega}_j$ with a face contained in $\Gamma$ and choose $\varphi = \tilde{\varphi} \in \widetilde{V}^h_j = V^h|_{\widetilde{\Omega}_j}$ in (5.5) such that, if $\widetilde{\Gamma}_{jk}$ is an interior face common with the element $\Omega_k$,

$$P_\tau \tilde{\varphi}(\xi_{jk}) = \begin{cases} \widetilde{\lambda}^h_{jk} & \text{on } \widetilde{\Gamma}_{jk}, \\ 0 & \text{on } \partial\Omega_j \setminus \widetilde{\Gamma}_{jk}. \end{cases}$$

Hence,

$$\langle\!\langle \widetilde{\lambda}^h_{jk}, \widetilde{\lambda}^h_{jk} \rangle\!\rangle_{\widetilde{\Gamma}_{jk}} = 0,$$

so that $\widetilde{\lambda}^h$ vanishes on $\widetilde{\Gamma}_{jk}$. The same argument shows that $\widetilde{\lambda}^h$ vanishes on all interior faces of $\widetilde{\Omega}_j$. Next, let $\widetilde{\Omega}_k$ be an interior element with a face $\widetilde{\Gamma}_{jk}$ common with a boundary element $\Omega_j$, so that $\widetilde{\lambda}_{jk} = -\widetilde{\lambda}_{kj} = 0$ on $\widetilde{\Gamma}_{jk}$. For the other interior faces of $\widetilde{\Omega}_k$, we can repeat the argument given for the boundary elements to show that $\widetilde{\lambda}^h$ vanishes on all interior faces of $\widetilde{\Omega}_k$. In this way we proceed until the domain is exhausted. Thus, we have proved the following theorem.

**Theorem 5.1** *Problem* (5.1) *has a unique solution. Moreover,* $(\widetilde{E}^h, \widetilde{H}^h)$ *is a solution of* (3.7) *and satisfies the estimates proved in Theorem 4.1.*

## 6. The Domain-Decomposition Iterative Procedure

We shall restrict the analysis to the case in which the partition $\mathcal{T}^h$ associated with the spaces $V^h \times W^h$ coincides with the domain decomposition partition. Then, set $F_j = F|_{\Omega_j}$, $G_j = G|_{\Omega_j}$ and consider the following decomposition of Problem (2.3)-(2.4): for $j = 1, \cdots, J$, find $(E_j, H_j)$ such that

$$\sigma E_j - \nabla \times H_j = F_j \qquad \text{in } \Omega_j, \tag{6.1a}$$

$$i\omega\mu H_j + \nabla \times E_j = G_j \qquad \text{in } \Omega_j, \tag{6.1b}$$

$$(1-i)P_\tau a E_j + \nu_j \times H_j = 0 \quad \text{on } \Gamma_j, \tag{6.1c}$$

subject to the interface consistency conditions

$$\nu_j \times H_j = -\nu_k \times H_k \quad \text{on } \Gamma_{jk}, \tag{6.2a}$$

$$P_\tau E_j = P_\tau E_k \qquad \text{on } \Gamma_{jk}. \tag{6.2b}$$

Instead of (6.2), we shall impose the Robin-type transmission conditions

$$(\nu_j \times H_j + \beta_{jk} P_\tau E_j) = -(\nu_k \times H_k - \beta_{jk} P_\tau E_k) \quad \text{on } \Gamma_{jk} \subset \partial\Omega_j, \tag{6.3a}$$

$$(\nu_k \times H_k + \beta_{jk} P_\tau E_k) = -(\nu_j \times H_j - \beta_{jk} P_\tau E_j) \quad \text{on } \Gamma_{kj} \subset \partial\Omega_k; \tag{6.3b}$$

here, $\beta_{jk}$ is a complex function defined on $\cup_{jk}\Gamma_{jk}$ with positive real and nonpositive imaginary parts.

The differential nonconforming domain decomposition is to find $(E_j, H_j) \in H^*(curl; \Omega_j) \times [L^2(\Omega_j)]^3$, $j = 1, \cdots, J$, such that

$$\begin{aligned}(\sigma E_j, \varphi)_j - (H_j, \nabla \times \varphi)_j + \sum_k \langle \beta_{jk}(P_\tau E_j - P_\tau E_k) + \nu_k \times H_k, P_\tau \varphi\rangle_{\Gamma_{jk}} \\ +(1-i)\langle P_\tau a E_j, P_\tau \varphi\rangle_{\Gamma_j} = (F_j, \varphi)_j, \quad \varphi \in H^*(curl; \Omega_j),\end{aligned} \tag{6.4a}$$

$$i\omega(\mu H_j, \psi)_j + (\nabla \times E_j, \psi)_j = (G_j, \psi)_j, \quad \psi \in [L^2(\Omega_j)]^3. \tag{6.4b}$$

To define an iterative procedure for the discrete problem as motivated by (6.4), introduce a new set,

$$\Lambda^h = \{\lambda^h : \lambda^h|_{\Gamma_{jk}} \equiv \lambda^h_{jk} \in \Lambda_{jk}, \ \forall \{j,k\}\}, \quad \Lambda_{jk} = P_0(\Gamma_{jk}) \times P_0(\Gamma_{jk}),$$

of Lagrange multipliers associated with $(\nu_j \times H_j)(\xi_{jk})$ on $\Gamma_{jk}$. Note that two copies, $\Lambda_{jk}$ and $\Lambda_{kj}$, of constant vector functions exist on $\Gamma_{jk}$. Then, choose an initial guess

$$(E_j^{h,0}, H_j^{h,0}, \lambda_{jk}^{h,0}, \lambda_{kj}^{h,0}) \in V_j^h \times W_j^h \times \Lambda_{jk} \times \Lambda_{kj}.$$

Then, find $(E_j^{h,n}, H_j^{h,n}, \lambda_{jk}^{h,n}) \in V_j^h \times W_j^h \times \Lambda_{jk}$ as the solution of the equations

$$\begin{aligned}(\sigma E_j^{h,n}, \varphi)_j - (H_j^{h,n}, \nabla \times \varphi)_j + \sum_k \langle\!\langle \beta_{jk} P_\tau E_j^{h,n}, P_\tau \varphi\rangle\!\rangle_{\Gamma_{jk}} \\ +(1-i)\langle\!\langle P_\tau a E_j^{h,n}, P_\tau \varphi\rangle\!\rangle_{\Gamma_j} \\ = (F_j, \varphi)_j + \sum_k \langle\!\langle \beta_{jk} P_\tau E_k^{h,n-1} - \lambda_{kj}^{h,n-1}, P_\tau \varphi\rangle\!\rangle_{\Gamma_{jk}}, \qquad \varphi \in V_{jk}^h,\end{aligned} \tag{6.5a}$$

$$i\omega(\mu H_j^{h,n}, \psi)_j + (\nabla \times E_j^{h,n}, \psi)_j = (G_j, \psi)_j, \qquad \psi \in W_j^h, \tag{6.5b}$$

$$\lambda_{jk}^{h,n} = -\lambda_{kj}^{h,n-1} + \beta_{jk}(P_\tau E_k^{h,n-1} - P_\tau E_j^{h,n})(\xi_{jk}), \qquad \xi_{jk} \in \Gamma_{jk}. \tag{6.5c}$$

Since $\beta_{jk}$ has positive real and nonpositive imaginary parts, these local problems are easily seen to be uniquely solvable.

We shall demonstrate the convergence of the iteration by showing that

$$(E_j^{h,n}, H_j^{h,n}, \lambda_{jk}^{h,n}) \longrightarrow (\widetilde{E}_j^h, \widetilde{H}_j^h, \widetilde{\lambda}_{jk}^h),$$

where $\widetilde{E}_j^h = \widetilde{E}^h|_{\Omega_j}$, $\widetilde{H}_j^h = \widetilde{H}^h|_{\Omega_j}$, and $\widetilde{\lambda}_{jk}^h = \widetilde{\lambda}^h|_{\Gamma_{jk}}$ and $(\widetilde{E}^h, \widetilde{H}^h, \widetilde{\lambda}^h)$ satisfies (5.1). This result, combined with those stated in Theorem 4.1 and Theorem 5.1, will imply the convergence of the iterative procedure to the solution $(E, H)$ of Problem (2.3) or (2.6).

First, note that $(\widetilde{E}_j^h, \widetilde{H}_j^h)$ satisfies the local equation

$$\begin{aligned}(\sigma\widetilde{E}_j^h, \varphi)_j - (\widetilde{H}_j^h, \nabla\times\varphi)_j - \sum_k \langle\langle \widetilde{\lambda}_{jk}^h, P_\tau\varphi\rangle\rangle_{\Gamma_{jk}} + (1-i)\langle\langle P_\tau a\widetilde{E}_j^h, P_\tau\varphi\rangle\rangle_{\Gamma_j} \\ = (F_j, \varphi), \qquad \varphi \in V_j^h, \qquad (6.6a)\\ i\omega(\mu\widetilde{H}_j^h, \psi)_j + (\nabla\times\widetilde{E}_j^h, \psi)_j = (G_j, \psi), \quad \psi \in W_j^h. \qquad (6.6b)\end{aligned}$$

Also, since $\widetilde{\lambda}_{jk}^h = -\widetilde{\lambda}_{kj}^h$, (5.1c) is equivalent to

$$\widetilde{\lambda}_{jk}^h = -\widetilde{\lambda}_{kj}^h + \beta_{jk}(P_\tau\widetilde{E}_k^h - P_\tau\widetilde{E}_j^h)(\xi_{jk}), \quad \xi_{jk} \in \Gamma_{jk}. \qquad (6.7)$$

We shall restrict the convergence proof to the case $\beta_{jk} = \beta = \beta_R - i\beta_I$ with positive $\beta_R$ and $\beta_I$; the general case is easily treated similarly. The errors,

$$u_j^n = E_j^{h,n} - \widetilde{E}_j^h, \quad v_j^n = H_j^{h,n} - \widetilde{H}_j, \quad \theta_{jk}^n = \lambda_{jk}^{h,n} - \widetilde{\lambda}_{jk}^h,$$

satisfy

$$\begin{aligned}(\sigma u_j^n, \varphi)_j - (v_j^n, \nabla\times\varphi)_j - \sum_k \langle\langle \theta_{jk}^n, P_\tau\varphi\rangle\rangle_{\Gamma_{jk}} \\ +(1-i)\langle\langle P_\tau a u_j^n, P_\tau\varphi\rangle\rangle_{\Gamma_j} = 0, \quad \varphi \in V_j^h, \qquad (6.8a)\\ i\omega(\mu v_j^n, \psi)_j + (\nabla\times u_j^n, \psi)_j = 0, \quad \psi \in W_j^h, \qquad (6.8b)\\ \theta_{jk}^n = -\theta_{kj}^{n-1} + \beta(P_\tau u_k^{n-1} - P_\tau u_j^n)(\xi_{jk}), \quad \xi_{jk} \in \Gamma_{jk}. \qquad (6.8c)\end{aligned}$$

Choose $\varphi = u_j^n$ in (6.8a) and $\psi = v_j^n$ in the conjugate of (6.8b), and then add the resulting equations to obtain

$$(\sigma u_j^n, u_j^n) - i\omega(\mu v_j^n, v_j^n) - \sum_k \langle\langle \theta_{jk}^n, P_\tau u_j^n\rangle\rangle_{\Gamma_{jk}} + (1-i)\langle\langle P_\tau a u_j^n, P_\tau u_j^n\rangle\rangle_{\Gamma_j} = 0. \qquad (6.9)$$

Taking the real and imaginary parts in (6.9) gives

$$(\sigma u_j^n, u_j^n) - \mathrm{Re}\sum_k \langle\langle \theta_{jk}^n, P_\tau u_j^n\rangle\rangle_{\Gamma_{jk}} + \langle\langle P_\tau a u_j^n, P_\tau u_j^n\rangle\rangle_{\Gamma_j} = 0, \qquad (6.10a)$$

$$\omega(\mu v_j^n, v_j^n) + \mathrm{Im}\sum_k \langle\langle \theta_{jk}^n, P_\tau u_j^n\rangle\rangle_{\Gamma_{jk}} + \langle\langle P_\tau a u_j^n, P_\tau u_j^n\rangle\rangle_{\Gamma_j} = 0. \qquad (6.10b)$$

Since $|p \pm \beta q|^2 = |p|^2 + |\beta|^2|q|^2 \pm 2\beta_R \mathrm{Re}(p\overline{q}) \mp 2\beta_I \mathrm{Im}(p\overline{q})$, (6.10) implies that

$$\begin{aligned}\sum_{jk} |\theta_{jk}^n \pm \beta P_\tau u_j^n(\xi_{jk})|_{0,\Gamma_{jk}}^2 = \sum_{jk} |\theta_{jk}^n|_{0,\Gamma_{jk}}^2 + |\beta|^2 |P_\tau u_j^n(\xi_{jk})|_{0,\Gamma_{jk}}^2 \\ \pm 2\beta_R \sum_j [(\sigma u_j^n, u_j^n) + \langle\langle P_\tau a u_j^n, P_\tau u_j^n \rangle\rangle_{\Gamma_j}] \\ \pm 2\beta_I \sum_j [\omega(\mu v_j^n, v_j^n)_j + \langle\langle P_\tau a u_j^n, P_\tau u_j^n \rangle\rangle_{\Gamma_j}]. \qquad (6.11)\end{aligned}$$

Set

$$R^n = R(u^n, v^n, \theta^n) = \sum_{jk} |\theta_{jk}^n + \beta P_\tau u_j^n(\xi_{jk})|_{0,\Gamma_{jk}}^2, \quad n \geq 1. \qquad (6.12)$$

Then, by (6.8c) and (6.11),

$$\begin{aligned} R^n &= \sum_{jk} |\theta_{kj}^{n-1} - \beta P_\tau u_k^{n-1}(m_{kj})|_{0,\Gamma_{kj}}^2 \\ &= R^{n-1} - 4\beta_R \sum_k \left[(\sigma u_k^{n-1}, u_k^{n-1})_k + \langle\langle P_\tau a u_k^{n-1}, P_\tau u_k^{n-1} \rangle\rangle_{\Gamma_k \cap \Gamma}\right] \\ &\quad -4\beta_I \sum_k \left[\omega(\mu v_k^{n-1}, v_k^{n-1})_k + \langle\langle P_\tau a u_k^{n-1}, P_\tau u_k^{n-1} \rangle\rangle_{\Gamma_k \cap \Gamma}\right]. \qquad (6.13)\end{aligned}$$

Let

$$T_{F,G} : \mathcal{N}C_{-1}^h \times W^h \times \Lambda^h \to \mathcal{N}C_{-1}^h \times W^h \times \Lambda^h$$

be the affine map such that $(E, H, \lambda) \equiv T_{F,G}(U, V, \theta)$ is the solution of the equations

$$\begin{aligned}(\sigma E_j, \varphi)_j - (H_j, \nabla \times \varphi)_j + \sum_k \langle\langle \beta_{jk} P_\tau E_j, P_\tau \varphi \rangle\rangle_{\Gamma_{jk}} + (1-i)\langle\langle P_\tau a E_j, P_\tau \varphi \rangle\rangle_{\Gamma_j} \\ = (F_j, \varphi) + \sum_k \langle\langle \beta_{jk} P_\tau U_k - \theta_{kj}, P_\tau \varphi \rangle\rangle_{\Gamma_{jk}}, \qquad \varphi \in V_j^h, \qquad (6.14a)\end{aligned}$$

$$i\omega(\mu H_j, \psi)_j + (\nabla \times E_j, \psi)_j = (G_j, \psi), \qquad \psi \in W_j^h, \qquad (6.14b)$$

$$\lambda_{jk} = -\theta_{kj} + \beta_{jk}(P_\tau U_k - P_\tau E_j)(\xi_{jk}), \qquad \xi_{jk} \in \Gamma_{jk}. \qquad (6.14c)$$

**Lemma 6.1** *If $(E, H, \lambda)$ is a fixed point of $T_{F,G}$, then $\lambda_{jk} = -\lambda_{kj}$ for all $\{j, k\}$. Moreover, the triple $(E, H, \lambda)$ is a solution of (6.6)–(6.7) if and only if it is a fixed point of $T_{F,G}$.*

**Proof.** Let $(E, H, \lambda)$ be a fixed point of $T_{F,G}$. By (6.14c),

$$\lambda_{jk} = -\lambda_{kj} + \beta_{jk}(P_\tau E_k - P_\tau E_j)(\xi_{jk}), \quad \lambda_{kj} = -\lambda_{jk} + \beta_{jk}(P_\tau E_j - P_\tau E_k)(\xi_{jk}).$$

Combining these equations gives

$$P_\tau E_j(\xi_{jk}) = P_\tau E_k(\xi_{jk}) \text{ and } \lambda_{jk} = -\lambda_{kj}.$$

Thus, any fixed point of $T_{F,G}$ is a solution of (6.6)-(6.7), and $\lambda_{jk} = -\lambda_{kj}$ for any fixed point. The second part of our assertion is trivial. This completes the proof.

Note that $T_{F,G}(U,V,\theta) = T_0(U,V,\theta) + T_{F,G}(0,0,0)$, $T_0$ being $T_{F,G}$ with $F = G = 0$, and $(U,V,\theta)$ is a fixed point of $T_{F,G}$ if and only if $(U,V,\theta) = T_0(U,V,\theta) + T_{F,G}(0,0,0)$, so that a fixed point $(U,V,\theta)$ of $T_{F,G}$ is a solution of

$$(I - T_0)(U,V,\theta) = T_{F,G}(0,0,0).$$

Set

$$h_{\max}(\Omega_j) = \max(h_x(\Omega_j), h_y(\Omega_j), h_z(\Omega_j)), \qquad h_{\max} = \max_j h_{\max}(\Omega_j),$$
$$h_{\min}(\Omega_j) = \min(h_x(\Omega_j), h_y(\Omega_j), h_z(\Omega_j)), \qquad h_{\min} = \min_j h_{\min}(\Omega_j).$$

We have the following estimate of the spectral radius for the iterative procedure (6.5).

**Theorem 6.1** *The spectral radius $\rho(T_0)$ of $T_0$ is less than one for any choice of $\beta = \beta_0(1-i)$, $\beta_0 > 0$. For $\beta$ given by (6.26),*

$$\rho(T_0) \le 1 - Ch_{\min}.$$

**Proof.** Let $\gamma$ be an eigenvalue of $T_0$ and let $(E,H,\lambda)$ be an associated eigenvector, so that

$$T_0(E,H,\lambda) = \gamma(E,H,\lambda). \tag{6.15}$$

It follows from (6.12) that

$$R(T_0(E,H,\lambda)) = |\gamma|^2 R(E,H,\lambda), \tag{6.16}$$

and, from (6.13),

$$\begin{aligned} R(T_0(E,H,\lambda)) \quad &= \quad R(E,H,\lambda) - 4\beta_R \sum_j [(\sigma E_j, E_j)_j + \langle\langle P_\tau a E_j, P_\tau E_j\rangle\rangle_{\Gamma_j}] \\ &\qquad -4\beta_I \sum_j [\omega(\mu H_j, H_j)_j + \langle\langle P_\tau a E_j, P_\tau E_j\rangle\rangle_{\Gamma_j}]. \end{aligned} \tag{6.17}$$

Combining (6.16) and (6.17) leads to

$$\begin{aligned} |\gamma|^2 \quad &= \quad \Big\{ R(E,H,\lambda) - 4\beta_R \sum_j [(\sigma E_j, E_j)_j + \langle\langle P_\tau a E_j, P_\tau E_j\rangle\rangle_{\Gamma_j}] \\ &\qquad -4\beta_I \sum_j [\omega(\mu H_j, H_j) + \langle\langle P_\tau a E_j, P_\tau E_j\rangle\rangle_{\Gamma_j}] \Big\} \Big/ R(E,H,\lambda). \end{aligned} \tag{6.18}$$

Hence, $|\gamma| \le 1$.

Assume that we can show that, for any eigenvector $(E,H,\lambda)$ of $T_0$,

$$\begin{aligned} R(E,H,\lambda) \quad &\le \quad 4M\beta_R \sum_j \left[(\sigma E_j, E_j)_j + \langle\langle P_\tau a E_j, P_\tau E_j\rangle\rangle_{\Gamma_j}\right] \\ &\qquad +4M\beta_I \sum_j \left[\omega(\mu H_j, H_j)_j + \langle\langle P_\tau a E_j, P_\tau E_j\rangle\rangle_{\Gamma_j}\right]. \end{aligned} \tag{6.19}$$

Then, combining (6.18) and (6.19) would imply that

$$|\gamma|^2 \leq 1 - \frac{1}{M}, \tag{6.20}$$

as desired. Also, we will see later that (6.20) will yield an estimate on the rate of convergence of (6.5) in terms of the sizes and shapes of the subdomains $\Omega_j$ of the domain decomposition partition.

Thus, we consider the validity of (6.19). If $\gamma = 0$, (6.20) is trivial. Therefore, we only have to prove (6.19) for $\gamma \neq 0$. By (6.15), (6.14c), and (6.14a),

$$\begin{aligned} (\sigma E_j, \varphi)_j - (H_j, \nabla \times \varphi)_j - \sum_k \langle\langle \lambda_{jk}, P_\tau \varphi \rangle\rangle_{\Gamma_{jk}} & \\ +(1-i)\langle\langle P_\tau a E_j, P_\tau \varphi \rangle\rangle_{\Gamma_j} = 0, & \qquad \varphi \in V_j^h, \qquad (6.21a)\\ i\omega(\mu H_j, \psi)_j + (\nabla \times E_j, \psi)_j = 0, & \qquad \psi \in W_j^h. \qquad (6.21b) \end{aligned}$$

Let $\widetilde{\Omega}_j$ be an arbitrary element and let $\widetilde{\Gamma}_{jk}$ be an interior face of $\widetilde{\Omega}_j$ common with another element $\Omega_k$. Choose $\varphi = \widetilde{\varphi} \in V_j^h$ in (6.21a) such that

$$P_\tau \widetilde{\varphi}(\xi_{jk}) = \begin{cases} \widetilde{\lambda}_{jk} & \text{on } \widetilde{\Gamma}_{jk}, \\ 0 & \text{on } \partial\Omega_j \setminus \widetilde{\Gamma}_{jk}. \end{cases}$$

By scaling arguments,

$$\|\widetilde{\varphi}\|_{0,\Omega_j}^2 \leq C_1 \langle\langle \widetilde{\lambda}_{jk}, \widetilde{\lambda}_{jk} \rangle\rangle_{\widetilde{\Gamma}_{jk}}, \quad \|\nabla \times \widetilde{\varphi}\|_{0,\Omega_j}^2 \leq C_2 \langle\langle \widetilde{\lambda}_{jk}, \widetilde{\lambda}_{jk} \rangle\rangle_{\widetilde{\Gamma}_{jk}}, \tag{6.22}$$

where

$$C_1 = C \max_j \left\{ h_{\max}(\Omega_j)^2 / h_{\min}(\Omega_j) \right\}, \quad C_2 = C \max_j \left\{ h_{\max}(\Omega_j)^2 / h_{\min}(\Omega_j)^3 \right\}.$$

Thus, by (6.21a),

$$\begin{aligned} \langle\langle \widetilde{\lambda}_{jk}, \widetilde{\lambda}_{jk} \rangle\rangle_{\widetilde{\Gamma}_{jk}} &= (\sigma E_j, \widetilde{\varphi})_j - (H_j, \nabla \times \widetilde{\varphi})_j \\ &\leq C \left[ \sqrt{C_1} \|E_j\|_{0,\Omega_j} + \sqrt{C_2} \|H_j\|_{0,\Omega_j} \right] \langle\langle \widetilde{\lambda}_{jk}, \widetilde{\lambda}_{jk} \rangle\rangle_{\widetilde{\Gamma}_{jk}}^{1/2}, \end{aligned}$$

and we can conclude that, for all elements $\Omega_j$,

$$\sum_k \langle\langle \lambda_{jk}, \lambda_{jk} \rangle\rangle_{\Gamma_{jk}} \leq C \left[ C_1 \|E_j\|_{0,\Omega_j}^2 + C_2 \|H_j\|_{0,\Omega_j}^2 \right]. \tag{6.23}$$

Again, by scaling,

$$\langle\langle P_\tau E_j, P_\tau E_j \rangle\rangle_{\Gamma_{jk}} \leq C_3 \|E_j\|_{0,\Omega_j}^2, \quad C_3 = C \max_j \left\{ h_{\max}(\Omega_j) / h_{\min}(\Omega_j)^2 \right\}. \tag{6.24}$$

In what follows, let $\beta = \beta_0(1-i)$, $\beta_0 > 0$. Also, let $C$ denote a generic positive constant. By invoking (6.10), (6.12), and (6.17), a combination of (6.23) and (6.24) gives

$$\begin{aligned} R(E,H,\lambda) &= \sum_{jk} |\lambda_{jk} + \beta P_\tau E_j(\xi_{jk})|^2_{0,\Gamma_{jk}} \\ &= \sum_{jk} \Big\{ |\lambda_{jk}|^2_{0,\Gamma_{jk}} + 2\beta_0^2 \langle\!\langle P_\tau E_j, P_\tau E_j \rangle\!\rangle_{\Gamma_{jk}} \\ &\qquad +2\beta_0 \mathrm{Re}\langle\!\langle \lambda_{jk}, P_\tau E_j\rangle\!\rangle_{\Gamma_{jk}} - 2\beta_0 \mathrm{Im}\langle\!\langle \lambda_{jk}, P_\tau E_j\rangle\!\rangle_{\Gamma_{jk}} \Big\} \\ &\le C \sum_j \Big\{ C_1 \|E_j\|^2_{0,\Omega_j} + C_2 \|H_j\|^2_{0,\Omega_j} + \beta_0^2 C_3 \|E_j\|^2_{0,\Omega_j} \Big\}. \end{aligned} \tag{6.25}$$

Since it was assumed that $\sigma \ge \sigma_0 > 0$ and $\mu \ge \mu_0 > 0$,

$$\begin{aligned} R(E,H,\lambda) &\le C \sum_j \Big\{ \frac{C_1}{\sigma_0}(\sigma E_j, E_j)_j + \frac{C_2}{\omega\mu_0}\omega(\mu H_j, H_j)_j + \beta_0^2 \frac{C_3}{\sigma_0}(\sigma E_j, E_j)_j \Big\} \\ &\le 4M(\beta_0)\beta_0 \sum_j \Big\{ (\sigma E_j, E_j)_j + \omega(\mu H_j, H_j)_j + \langle\!\langle P_\tau a E_j, P_\tau E_j\rangle\!\rangle_{\Gamma_j} \Big\}, \end{aligned}$$

where

$$M(\beta_0) = C\left( \frac{C_1}{\sigma_0\beta_0} + \frac{C_2}{\omega\mu_0\beta_0} + \frac{C_3\beta_0}{\sigma_0} \right).$$

As a function of $\beta_0$, $M(\beta_0)$ is minimized by the choice

$$\beta_0^* = \sqrt{\left( \frac{C_1}{\sigma_0} + \frac{C_2}{\omega\mu_0} \right) \frac{\sigma_0}{C_3}}\,; \tag{6.26}$$

for this choice, with $\zeta = \max_j [h_{\max}(\Omega_j)/h_{\min}(\Omega_j)]$, one has

$$M(\beta_0^*) = C\sqrt{\left( \frac{C_1}{\sigma_0} + \frac{C_2}{\omega\mu_0} \right) \frac{C_3}{\sigma_0}} \sim C\zeta^{3/2}/h_{\min},$$

so that $|\gamma|^2 \le 1 - C\zeta^{-3/2} h_{\min}$. Thus, it follows that

$$\rho(T_0) \le 1 - C\zeta^{-3/2} h_{\min}, \tag{6.27}$$

which demonstrates (6.20) and completes the proof.

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## References

1. D. N. Arnold and F. Brezzi. *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates.* R.A.I.R.O. Modélisation Mathématique et Analyse Numérique, 19 (1985) 7-32.
2. L. Cagniard. *Basic theory of magneto-telluric method of geophysical prospecting.* Geophysics, 18 (1953) 605–635.
3. Ph. G. Ciarlet. The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam · New York · Oxford (1978).
4. J. H. Coggon. *Electromagnetic and electrical modelling by the finite element method.* Geophysics, 36 (1971) 132–155.
5. B. Després. *Méthodes de décomposition de domaines pour les problèmes de propagation d'ondes en régime harmonique.* Thèse, Université Paris IX Dauphine, UER Mathématiques de la Décision, 1991.
6. B. Després, P. Joly, and J. E. Roberts. *A domain decomposition method for the harmonic Maxwell equations.* In "Iterative Methods in Linear Algebra", R. Beauwens and P. de Groen, eds., pages 475–484. Elsevier Science Publishers B. V. (North-Holland), Amsterdam, 1992.
7. J. Douglas, Jr., P. J. Paes Leme, J. E. Roberts, and J. Wang. *A parallel iterative procedure applicable to the approximate solution of second order partial differential equations by mixed finite element methods.* Numer. Math., 65 (1993) 95–108.
8. J. Douglas, Jr., J. E. Santos, D. Sheen, and X. Ye. *Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems.* Mathematical Modelling and Numerical Analysis ($M^2AN$), 33 (1999) 747–770.
9. X. Feng. *Absorbing boundary conditions for electromagnetic wave propagation.* Math. Comp. 68 (1999) 145–168.
10. B. X. Fraeijs de Veubeke. *Displacement and equilibrium models in the finite element method.* In "Stress Analysis", O. C. Zienkiewicz and G. Holister, eds. Wiley, New York, 1965.
11. B. X. Fraeijs de Veubeke. *Stress function approach.* International Congress on the Finite Element Method in Structural Mechanics, Bournemouth, 1975.
12. P. L. Lions. *On the Schwarz alternating method I.* In "Domain Decomposition Methods for Partial Differential Equations", R. Glowinski, G. Golub, G. Meurant, and J. Periaux, eds., pages 1–42. SIAM, Philadelphia, 1988.
13. P. L. Lions. *On the Schwarz alternating method III: a variant for nonoverlapping subdomains.* In "Domain Decomposition Methods for Partial Differential Equations", T. F. Chan, R. Glowinski, J. Periaux, and O. B. Widlund, eds., pages 202–223. SIAM, Philadelphia, 1990.
14. R. L. Mackie, T. R. Madden, and P. E. Wannamaker. *Three-dimensional magnetotelluric modeling using difference equations — Theory and comparisons to integral equation solutions.* Geophysics, 58 (1993) 215–226.
15. P. B. Monk. *A mixed method for approximating Maxwell's equations.* SIAM J. Numer. Anal., 28 (1991) 1610–1634.
16. P. Monk. *A finite element method for approximating the time-harmonic Maxwell equations.* Numer. Math., 63 (1992) 243–261.
17. P. Monk. *A comparison of three mixed methods for the time dependent Maxwell's equations.* SIAM J. Sci. Stat. Comput., 13 (1992) 1097–1122.
18. P. Monk. *An analysis of Nédélec's method for the spatial discretization of Maxwell's equations.* J. Comp. and Appl. Math., 47 (1993) 101–121.
19. P. Monk and E. Süli. *A convergence analysis of Yee's scheme on nonuniform grids.* SIAM J. Numer. Anal., 31 (1994) 393–412.
20. J. C. Nedelec. *Mixed finite elements in* $\mathbb{R}^3$. Numer. Math., 35 (1980) 315–341.
21. J. E. Santos. *Global and domain-decomposed mixed methods for the solution of*

*Maxwell's equations with application to magnetotellurics.* Numerical Methods for Partial Differential Equations, 14 (1998), 407–437.
22. J. E. Santos and D. Sheen. *Global and parallelizable domain-decomposed mixed finite element methods for three-dimensional electromagnetic modelling.* Computational and Applied Mathematics, 17 (1998) 263–280.
23. J. E. Santos and D. Sheen. *On the existence and uniqueness of Maxwell's equations in bounded domains with application to magnetotellurics.* Math. Models Meth. Appl. Sci. 10 (2000)
24. D. Sheen. *A generalized Green's theorem.* Appl. Math. Lett., 5 (1992) 95–98.
25. D. Sheen. *Approximation of electromagnetic fields: Part I. Continuous problems.* SIAM J. Appl. Math., 57 (1997) 1716–1736.
26. B. J. Travis and A. D. Chave. *A moving finite element method for magnetotelluric modelling.* Physics of the Earth and Planetary Interiors, 53 (1989) 432–443.
27. P. E. Wannamaker and J. A. Stodt. *A stable finite element solution for two–dimensional magnetotelluric modelling.* Geophys. J. R. Astr. Soc., 88 (1987) 277–296.
28. K. Yee. *Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media.* IEEE Trans. Antennas and Propagation, AP-16 (1966) 302–307.